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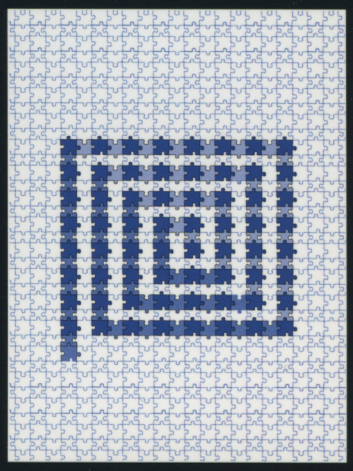
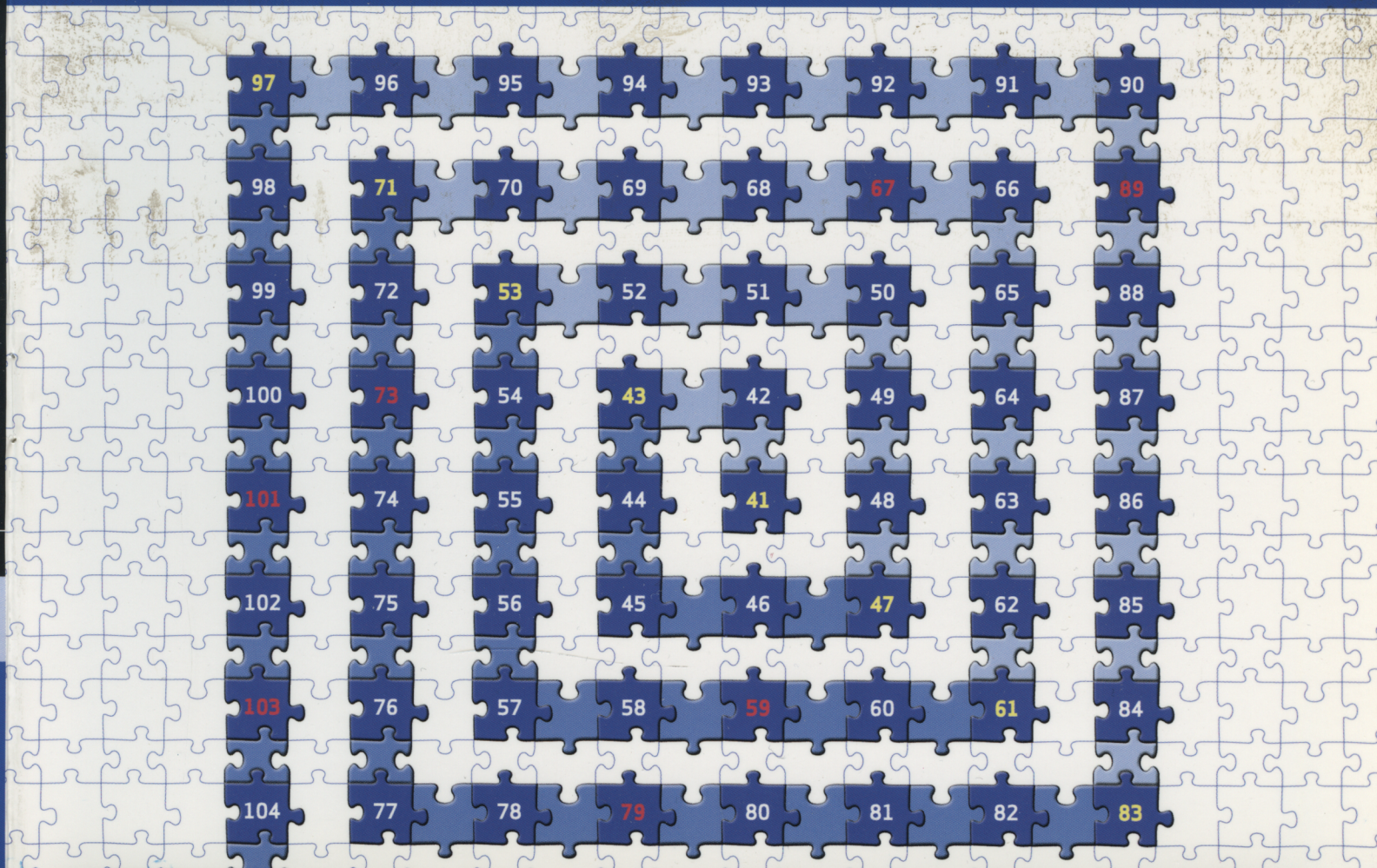
M381 Number Theory and Mathematical Logic



Number Theory **Unit 1**

Foundations



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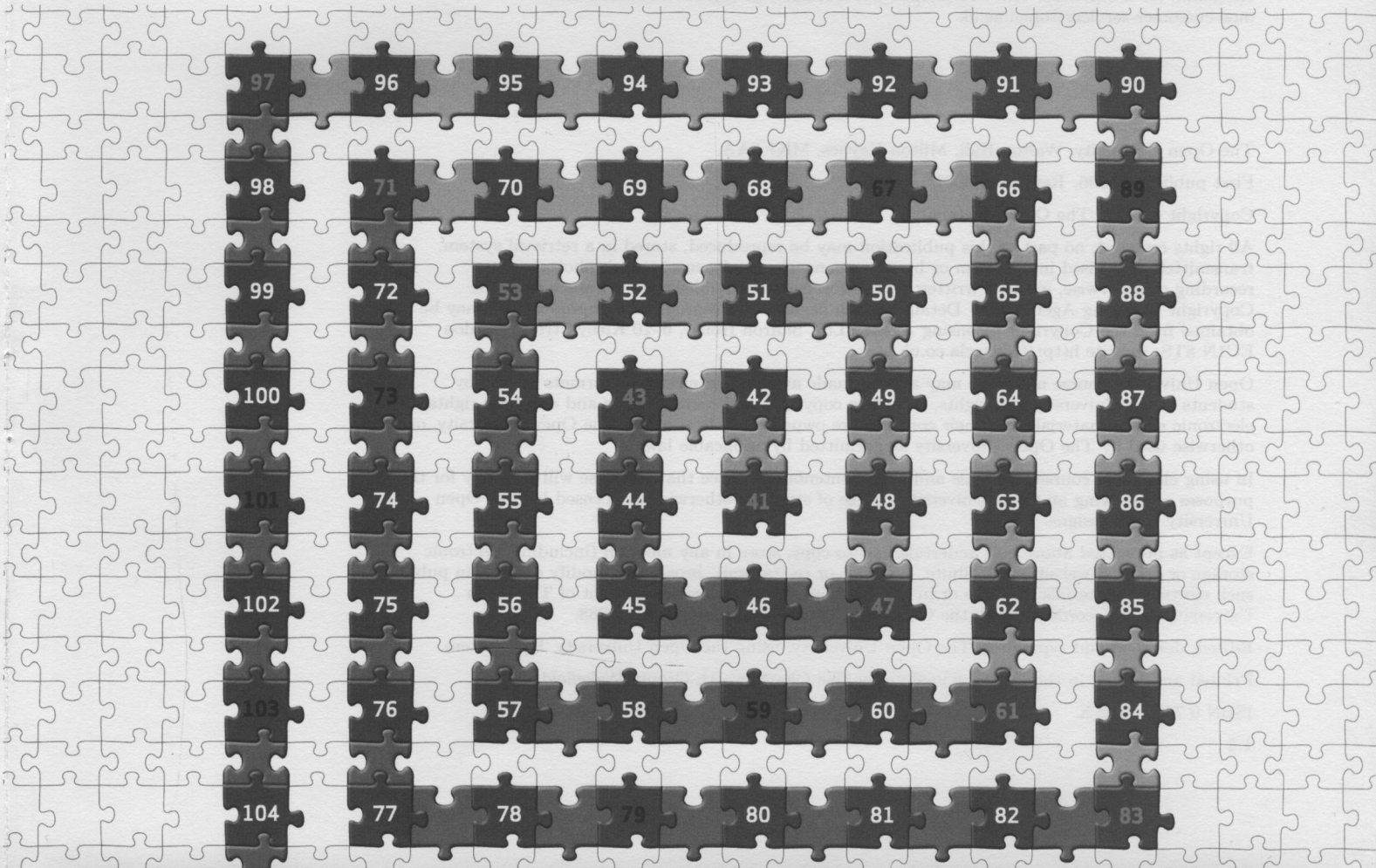
M381 Number Theory and Mathematical Logic

Number Theory **Unit 1**

Foundations

Prepared for the Course Team by Alan Best



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INTRODUCTION

The objective of this first unit is to lay a firm foundation for the whole course by examining in detail the fundamental ideas which form the basis from which to study the properties of numbers. We shall introduce, and make precise, the key notion of *divisibility*. For this we use a result concerning the division of one integer by another which will be familiar to you. The result is known as the *Division Algorithm*. Its consequences, both practical and theoretical, make it a cornerstone of number theory. Through it we shall embark on a study of the properties of divisibility, and we shall introduce the concept of the *greatest common divisor* — a topic which might already be familiar to you and one which is going to figure prominently in this course. The unit will also review a powerful method of proof in number theory: *Proof by Mathematical Induction*. You may well have met this idea before, but it is of such importance to us that we shall give a complete treatment here.

Before we get down to any theoretical ideas we need some families of numbers to which we might apply our techniques, and that is where we shall begin, after first giving a brief preview of some of the tasks which will confront us in the course.

1 WHAT IS NUMBER THEORY?

> The elementary theory of numbers should be one of the very best subjects for early mathematical instruction. It demands very little previous knowledge; its subject matter is tangible and familiar; the processes of reasoning it employs are simple, general and few; and it is unique among the mathematical sciences in its appeal to natural human curiosity. A month's intelligent instruction in the theory of numbers ought to be twice as instructive, twice as useful, and ten times more entertaining as the same amount of 'calculus for engineers'.
>
> G.H.Hardy, *Bulletin of the AMS* (1929).

There is an irresistible fascination in searching for numbers with specified properties. Nearly every century, as far back as the history of mathematics can be traced, has witnessed new and exciting discoveries concerning properties of numbers. Many of the greatest mathematicians, despite having their major interests elsewhere, have at some time in their careers been drawn into problems of number theory and have contributed to the body of knowledge. So what is the appeal of this subject both for professional mathematicians and for thousands of amateurs?

Consider the following problems:

1. Find all the divisors of 4 294 967 297.
2. A right-angled triangle has the property that all three of its sides have length equal to a whole number of units. If one of the sides is 24 find seven pairs of values for the other two sides.
3. Find positive integers x and y such that $x^2 - 41y^2 = 1$.
4. Show that

   $$1 \times 2 \times 3 \times 4 \times \cdots \times (n-1) + 1$$

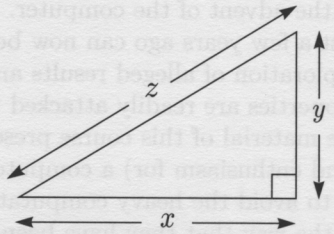
   is always divisible by n if n is prime, but is never divisible by n if n is composite.

5. Can every even integer from 4 onwards be expressed as the sum of two primes?
6. For which integers n is $2^n - 1$ a prime?

Each of these problems, which will confront us at some stage of this course, illustrates the most attractive feature of number theory; the problems can be understood by beginners of the subject. Perhaps you are not completely familiar with terms (which we shall explain later) such as prime, composite and divisor, but you should have a feeling for what is involved in these problems. Indeed, you could well take pencil and paper and start exploring some of them; there is no substantial body of knowledge needed as a prerequisite of becoming involved in number theory.

But while there is no difficulty posing these problems in a readily intelligible form, the varying degrees of difficulty in solving them highlights the most intriguing feature of number theory.

Because of the size of the number involved, Problem 1 is quite tricky. However any reader who is familiar with computers might quickly solve this problem. As it happens the smallest positive divisor of the given number is 641, a fact which, in the 17th century, eluded one of the all-time great mathematicians, Fermat, who had a particular interest in this very problem.



The Pythagorean equation $x^2 + y^2 = z^2$

Problem 2 asks for solutions in whole numbers of the famous equation of Pythagoras relating the edge lengths of a right-angled triangle, $x^2 + y^2 = z^2$. It shares a common feature with Problem 3 in that they both ask for integer (whole number) solutions of algebraic equations. Such equations are given their own name; they are called Diophantine equations (after the Greek mathematician, Diophantus, of the early Christian era). Many books have been written about Diophantine equations. What emerges is a lack of a general theory for solving them; each Diophantine equation appears to be a problem in its own right requiring its own method of solution. We shall meet many instances of Diophantine equations in this course, including the two posed above, which we shall show how to solve. In the meantime, if you are familiar with the 'smallest' solution in integers of the Pythagorean equation, namely $3^2 + 4^2 = 5^2$, then you may have spotted two of the solutions required by Problem 2 by scaling, namely $18^2 + 24^2 = 30^2$ and $24^2 + 32^2 = 40^2$. But what about the other five solutions? You might discover them by patient trial and error — but I doubt that you would solve Problem 3 in this way.

The smallest solution (of infinitely many) to Problem 3 is given by $x = 2049$, $y = 320$. The equation in Problem 3 is a particular case of the Pell equation, $x^2 - Ny^2 = 1$, named after the 17th century English mathematician John Pell. In the final unit of this course we shall find that we can solve Pell's equation for any value of N using just pencil and paper.

Problems 4, 5 and 6 are classics of number theory. Because the values involved in Problem 4 get very large, very quickly, it is not an easy problem to explore in depth by looking at special cases. But it turns out to be true, and we can give (and will do so in *Unit 4*) a general proof of this result. Problem 5 is easier to explore and, on finding it very straightforward to write each even integer up to 200 (say) as a sum of two primes, you would probably be tempted to conclude that it also is a true result. But, perhaps surprisingly, nobody has ever managed to prove it; to this day it remains an unsolved problem of mathematics despite assault by many great mathematicians. Problem 6 has its origin at the heart of another famous problem which we shall meet in *Unit 5*. By the year 1951 only 12 numbers n were known for which $2^n - 1$ is prime, but then electronic computers were brought to bear on the problem with the result that by the early 1990s thirty two such numbers were known. The largest one, $n = 756\,839$, led to the number $2^{756\,839} - 1$ being the largest known prime at that time. And yet, despite the scarcity of solutions to Problem 6, most mathematicians still

believe that there are infinitely many solutions to this problem, but proof of this seems, at the moment, hopelessly beyond reach.

Number theory is a subject which has developed over a long period of time and, as the previous paragraph suggests, is still very active today. We have already mentioned a few of the great mathematicians who contributed to the development of the subject; there are many, many more. In the course units we shall attempt to impart a flavour of the history of the subject. What we have written is by no means a systematic account of the history of number theory, and is not an assessed part of the course. Its inclusion will, we hope, enliven the theoretical side of the material and should also reveal the stumbling way in which progress has been made.

The nature of number theory has changed dramatically in recent years, due to the advent of the computer. Computations which were nigh impossible just a few years ago can now be managed with ease. Consequently exploration of alleged results and searches for numbers with prescribed properties are readily attacked with the assistance of a machine. Although the material of this course presents many challenges for those with access to (and enthusiasm for) a computer, we have written the course in such a way as to avoid the heavy computational side of the subject, attacking problems in the way that they have been tackled historically, using pencil and paper. There are, however, many problems in the course involving 'arithmetic' which could be simplified if you possess a calculator, and although it is not essential, it is advisable that you have one.

Whilst number theory is arguably the purest of all subjects within mathematics — little of what we shall meet in this course has immediate application elsewhere — you must not form the impression that it is all about development of theories and proof of results. On the contrary, number theory is about problem solving. Nobody becomes competent at calculus by reading about it; it is essential to practise differentiation and integration on a large number of examples. The same is true of number theory; you cannot get to grips with the subject without solving lots of problems yourself. To this end we have included a good stock of problems in all the units and while studying you should always have pencil and paper at hand. The *Examples* in the units are for you to read, but the *Problems* are for you to *do*. Do not spend forever on a problem which looks like defeating you. If you get stuck you should refer to the solution that is there — but do not give in too easily! *Additional Exercises* and their solutions are included at the end of each unit. It is not intended that you should necessarily work through all of these. If you feel confident that you have gained a thorough understanding of a certain section then there will be no need to attempt all the additional exercises on that section. On the other hand if a section has proved to be heavy going the additional exercises may prove to be a valuable source of help. At the end of the Additional Exercises section we have included a few *Challenge Problems*. These are intended only for the more able students who enjoy a challenge, since they are pitched at a level above the expected attainment for the course.

2 NUMBERS FROM PATTERNS

2.1 Polygonal numbers

Figure 2.1 illustrates the first five of the so-called *triangular numbers*, T_n. The arrangement of T_4 will be familiar to anyone who indulges in ten-pin bowling and T_5 is, more or less, the arrangement of the pack of 15 red balls at the start of a game of snooker. It is evident that the triangular numbers are the sums of consecutive integers starting from 1:

$T_1 = 1, \quad T_2 = 1+2, \quad T_3 = 1+2+3, \quad \ldots, \quad T_n = 1+2+3+\cdots+n.$

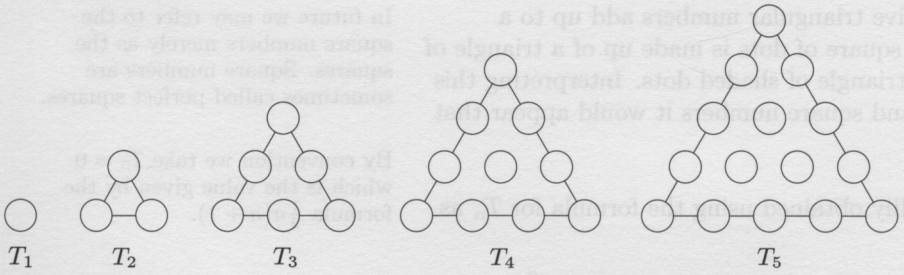


Figure 2.1 Triangular numbers

We can obtain a simple formula for T_n by writing the sum both forwards and backwards:

$$\begin{array}{rcccccccccc} T_n & = & 1 & + & 2 & + & 3 & + & \cdots & + & n \\ T_n & = & n & + & (n-1) & + & (n-2) & + & \cdots & + & 1 \end{array}$$

and adding. Each corresponding pair has a sum of $(n+1)$, so we obtain

$$2T_n = (n+1)+(n+1)+\cdots+(n+1) = n(n+1),$$

giving

$$T_n = 1+2+3+\cdots+n = \tfrac{1}{2}n(n+1).$$

This same argument can be used to find the sum of any *arithmetic series*. The general arithmetic series having n terms with first term a and common difference d is

$$a+(a+d)+(a+2d)+\cdots+(a+(n-1)d).$$

In this case, reversing and adding, each of the corresponding n pairs sum to $2a+(n-1)d$, and so we have

$$a+(a+d)+(a+2d)+\cdots+(a+(n-1)d) = \tfrac{1}{2}n(2a+(n-1)d).$$

We say that the terms of an arithmetic series,

$a, \ a+d, \ a+2d, \ \ldots,$

are in *arithmetic progression*.

This formula can be rewritten as

$n(a+\frac{1}{2}(n-1)d)$

illustrating that there are n terms of average $a+\frac{1}{2}(n-1)d$.

Problem 2.1

Determine the following sums.

(a) $3+10+17+24+\cdots+143$

(b) $100+98+96+\cdots+50$

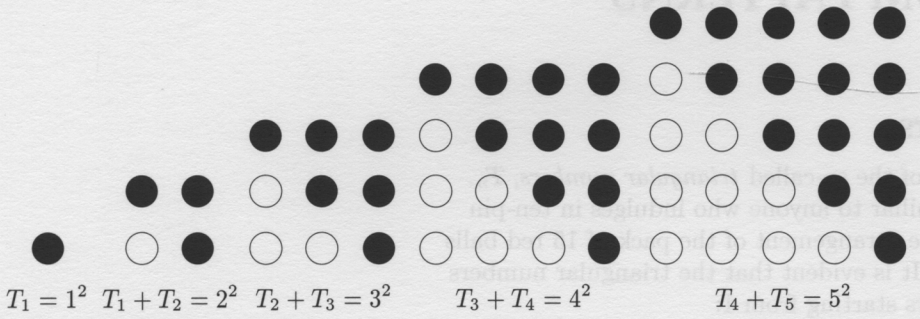


Figure 2.2 Adding consecutive triangular numbers

Figure 2.2 shows that two consecutive triangular numbers add up to a *square number*. In Figure 2.2, each square of dots is made up of a triangle of unshaded dots and the next larger triangle of shaded dots. Interpreting this observation in terms of triangular and square numbers it would appear that

$$T_{n-1} + T_n = n^2, \quad n \geq 1.$$

A formal proof of this result is readily obtained using the formula for T_n as follows.

$$T_{n-1} + T_n = \tfrac{1}{2}(n-1)n + \tfrac{1}{2}n(n+1) = \tfrac{1}{2}n(n-1+n+1) = n^2$$

In future we may refer to the square numbers merely as the squares. Square numbers are sometimes called perfect squares.

By convention we take $T_0 = 0$ which is the value given by the formula $\frac{1}{2}n(n+1)$.

Problem 2.2

Show that if n is a triangular number then so is $9n + 1$.

We may shorten the phrase 'is a triangular number' to 'is triangular'.

Problem 2.3

Show that an integer n is triangular if, and only if, $8n + 1$ is square.

Note that, since this is an 'if, and only if,' statement, it's proof will have to have two parts. You will need to show that, if n is triangular, then $8n + 1$ is square and, conversely, if $8n + 1$ is square, then n is triangular.

One instance of the result of Problem 2.3, is illustrated in Figure 2.3. A 9×9 square of dots is broken into eight copies of $T_4 = 10$ with one dot left over in the middle.

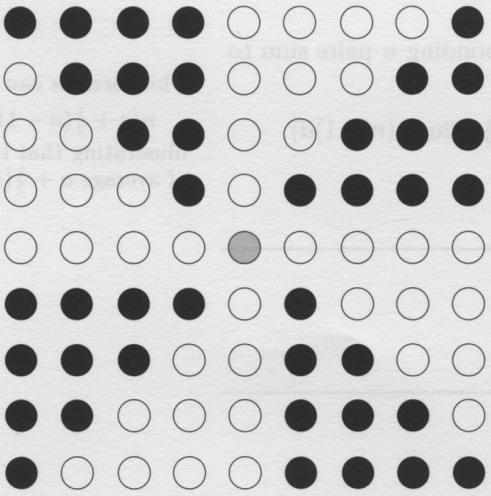


Figure 2.3 $8T_4 + 1 = 9^2$

While the triangular numbers are sums of consecutive integers,

$$1,\ 1+2,\ 1+2+3,\ \ldots,$$

Figure 2.4 illustrates that the squares are sums of consecutive odd integers,

$$1,\ 1+3,\ 1+3+5,\ \ldots;$$

the next odd number is added to the current square as dots along the bottom and right-hand side to produce the next square.

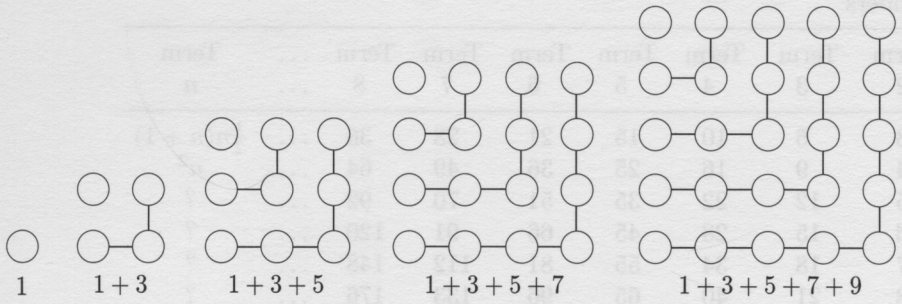


Figure 2.4 Squares as sums of odd integers

The general formula suggested by this sequence of diagrams is

$$1+3+5+\cdots+(2n-1)=n^2, \quad n\geq 1.$$

A proof of this result is immediate from the formula for an arithmetic series but we shall leave it aside for the time being. Formulae like this, which assert results for each positive integer n, are often tailor-made for proof by the method of Mathematical Induction which we deal with in Section 3.

We have seen that triangular numbers are sums of consecutive integers, and squares are sums of consecutive odd integers. The pattern can be continued: *pentagonal numbers* are the sums obtained from going up in threes, namely

$$1,\ 1+4,\ 1+4+7,\ 1+4+7+10,\ \ldots.$$

Figure 2.5 illustrates the start of the sequence of pentagonal numbers and Figure 2.6 goes one stage further and shows how the *hexagonal numbers*,

$$1,\ 1+5,\ 1+5+9,\ 1+5+9+13,\ \ldots,$$

arise from arrangements of dots forming hexagons.

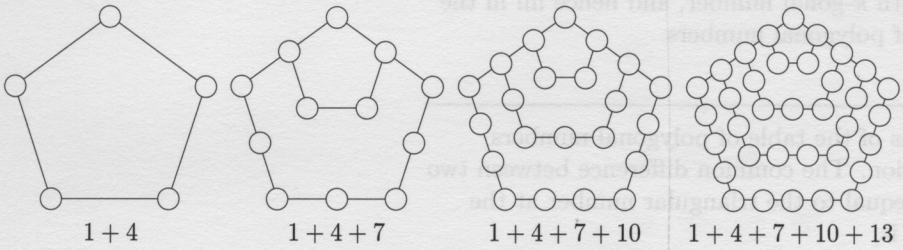


Figure 2.5 Pentagonal numbers

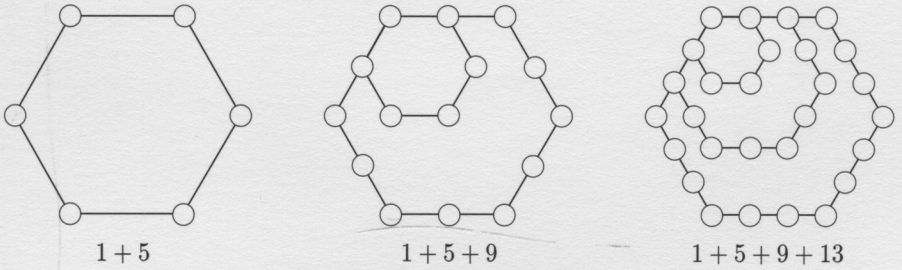


Figure 2.6 Hexagonal numbers

Below is a table showing the beginnings of the sequences of *polygonal numbers*. We have extended the table beyond numbers already seen by appealing to the emerging patterns. For example, the *octagonal numbers* form the sequence

$$1,\ 1+7,\ 1+7+13,\ 1+7+13+19,\ \ldots.$$

Table 2.1 Table of polygonal numbers

	Number of sides	Term 1	Term 2	Term 3	Term 4	Term 5	Term 6	Term 7	Term 8	...	Term n
Triangular	3	1	3	6	10	15	21	28	36	...	$\frac{1}{2}n(n+1)$
Square	4	1	4	9	16	25	36	49	64	...	n^2
Pentagonal	5	1	5	12	22	35	51	70	92	...	?
Hexagonal	6	1	6	15	28	45	66	91	120	...	?
Heptagonal	7	1	7	18	34	55	81	112	148	...	?
Octagonal	8	1	8	21	40	65	96	133	176	...	?
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
k-gonal	k	1	k	$3k-3$	?	?	?	?	?	...	$P(k,n)$

In the right-hand column we have listed the discovered formula for the nth triangular number and for the nth square. The rest are missing, but could be determined from the sum of the appropriate arithmetic series. Along the bottom row we have begun listing the terms for the k-gonal numbers. You may be able to spot formulae (in terms of k) for these numbers from the patterns that are present in the table.

Of course, we could readily fill in the right-hand column and the bottom row if we could find a general formula for the nth term in the sequence of k-gonal numbers, which we shall call $P(k,n)$. The next problem invites you to provide this formula. Remember that $P(3,n)$, the nth triangular number, is the sum of n terms of the arithmetic progression with first term 1 and common difference 1, $P(4,n)$, the nth square, is the sum of n terms of the arithmetic progression with first term 1 and common difference 2 and, in general, $P(k,n)$ is the sum of n terms of the arithmetic progression with first term 1 and common difference $k-2$.

Problem 2.4

Obtain a formula for $P(k,n)$, the nth k-gonal number, and hence fill in the missing entries in the above table of polygonal numbers.

Problem 2.5

The numbers in each of the columns of the table of polygonal numbers appear to be in arithmetic progression. The common difference between two successive terms in each column is equal to the triangular number at the head of the previous column. That is,

$$P(k,n) = P(k-1,n) + P(3,n-1).$$

Prove this formula algebraically.

2.2 Pyramidal numbers

The polygonal numbers arise as sums of arithmetic progressions with first term 1. We can take this a stage further: the *pyramidal numbers* are those which arise as sums of progressions of polygonal numbers with first term 1. For instance, from the sequence of triangular numbers,

$$1,\ 3,\ 6,\ 10,\ 15,\ \ldots,$$

we obtain the triangle-based pyramidal numbers as follows:

$$1,\ 1+3=4,\ 1+3+6=10,\ 1+3+6+10=20,\ \ldots.$$

Figure 2.7 illustrates how these numbers arise from layers of a triangle-based pyramid.

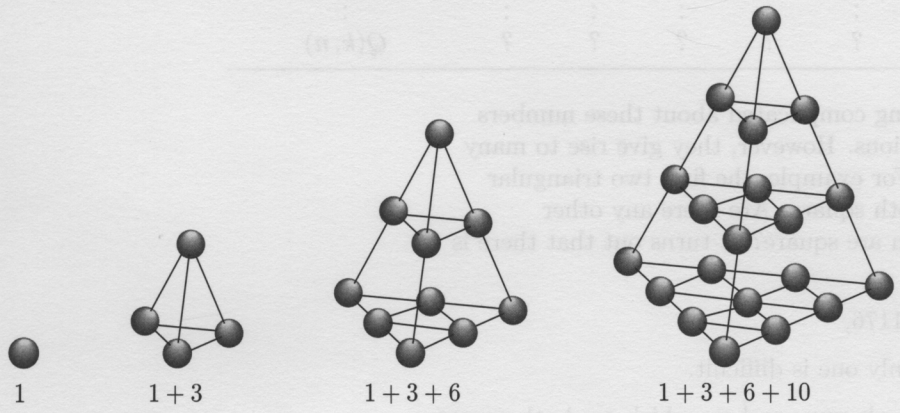


Figure 2.7 Triangle-based pyramidal numbers

Similarly, by adding consecutive terms of the sequence of squares we get the *square-based pyramidal numbers*

$$1,\ 1+4=5,\ 1+4+9=14,\ 1+4+9+16=30\ \ldots,$$

illustrated in Figure 2.8.

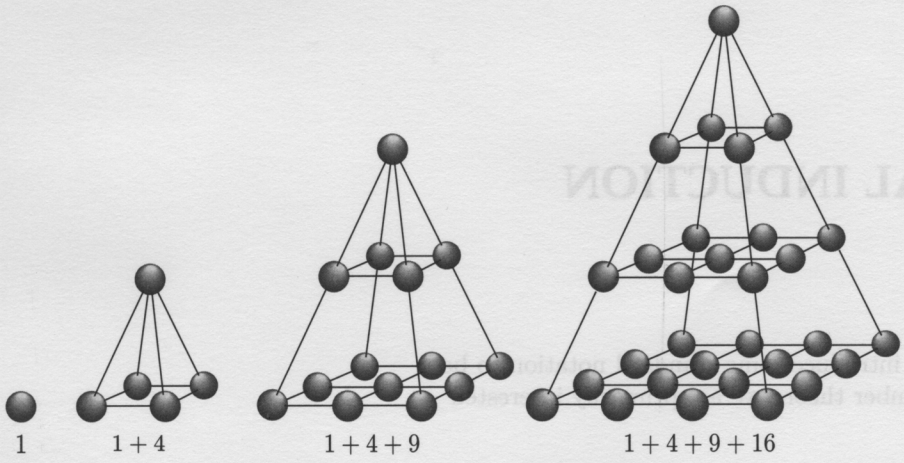


Figure 2.8 Square-based pyramidal numbers

The pyramidal numbers are readily generalized. For example, the heptagonal-based pyramidal numbers are generated as sums of heptagonal numbers,

$$1,\ 1+7=8,\ 1+7+18=26,\ 1+7+18+34=60,\ \ldots.$$

The following table gives some of the pyramidal numbers. We have started, but not completed, the row for the general k-gon and the column for the

general nth term. The formulae which occur in the latter are not so easily proved; we shall return to them when we have proof by Mathematical Induction in our armoury.

Table 2.2 Table of pyramidal numbers

Base	Term 1	Term 2	Term 3	Term 4	Term 5	Term 6	Term 7	Term 8	Term n
Triangle	1	4	10	20	35	56	84	120	$\frac{1}{6}n(n+1)(n+2)$
Square	1	5	14	30	55	91	140	204	$\frac{1}{6}n(n+1)(2n+1)$
Pentagon	1	6	18	40	75	126	196	288	?
Hexagon	1	7	22	50	95	161	252	372	?
Heptagon	1	8	26	60	115	196	308	456	?
Octagon	1	9	30	70	135	231	364	540	?
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
k-gon	1	$k+1$	$4k-2$	?	?	?	?	?	$Q(k,n)$

There may not appear to be anything complicated about these numbers derived from geometrical configurations. However, they give rise to many interesting and difficult problems. For example, the first two triangular pyramidal numbers, 1 and 4, are both square. Are there any other triangular pyramidal numbers which are square? It turns out that there is just one more:

$$140^2 = 1 + 3 + 6 + 10 + \cdots + 1176,$$

however the proof that this is the only one is difficult.

The table of polygonal numbers reveals two numbers which are both square and triangular, namely 1 and 36. Are there any more? As it happens the next one is 1225, followed by 41 616 and then 1 413 721. Problem 2.3 gives us a clue to one way of attacking this problem. The number x^2 will be triangular as well as square if, and only if, $8x^2 + 1$ is a square. In other words triangular squares will arise from integer solutions of the equation $8x^2 + 1 = y^2$. If x and y are integers satisfying this equation then x^2 is a triangular square. In fact there are infinitely many solutions of this equation, and hence infinitely many triangular squares. We shall see how to find them all later in the course.

3 MATHEMATICAL INDUCTION

3.1 Notation

Before progressing further we must introduce some standard notation to be used throughout this course. In number theory we are primarily interested in the *set of positive integers.*

The set of all integers, positive, negative or zero, is denoted by $\mathbb{Z}$:

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\},$$

and the set of positive integers by $\mathbb{Z}^+$:

$$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}.$$

The set of positive integers is sometimes called the set of natural numbers and denoted by $\mathbb{N}$.

We have described each of these sets by (partially) listing its elements enclosed in 'curly brackets' which are known more formally as braces. We

shall often describe a set by giving a property which characterizes its elements. For example, the set of all integers which are multiples of 3, and which is denoted by $3\mathbb{Z}$, can be described either by

$$3\mathbb{Z} = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$$

or by

$$3\mathbb{Z} = \{m : m = 3n, \text{ for some } n \in \mathbb{Z}\}.$$

We shall also have occasion to refer to the set of rational numbers $\mathbb{Q}$ and the set of real numbers $\mathbb{R}$. We shall make extensive use of the *modulus* or *absolute value* function which is defined for any element n of $\mathbb{Z}$ (or $\mathbb{Q}$ or $\mathbb{R}$) by

$$|n| = \begin{cases} n, & \text{if } n \geq 0, \\ -n, & \text{if } n < 0. \end{cases}$$

The rational numbers are the numbers $\frac{a}{b}$, where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}^+$.

Problem 3.1

(a) Which set of numbers is described by each of the following?

(i) $\{n \in \mathbb{Z} : |n| > 20\}$

(ii) $\{n : n = 2m^2, \text{ for some } m \in \mathbb{Z}^+\}$

(b) Describe each of the following sets in the notation used in part (a).

(i) The set of all odd positive integers.

(ii) The set of all integers lying between -100 and 100 inclusive.

3.2 Mathematical Induction

One property of $\mathbb{Z}^+$ which is not shared by some other number sets such as $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$ is worth recording. It may seem a rather obvious property, but its presence is crucial to establishing less apparent properties that we are going to need.

> ***The Well-Ordering Principle for $\mathbb{Z}^+$***
>
> Every non-empty subset of $\mathbb{Z}^+$ has a least member.
>
> That is, if S is a non-empty subset of $\mathbb{Z}^+$ then there exists $b \in S$ such that $b \leq n$ for all $n \in S$.

In future we shall refer to this just as the Well-Ordering Principle

This property says that any collection of positive integers that we care to describe, as long as it has some elements in it, must have a least member. Notice that the set of positive real numbers does not have this property. For example the set of positive reals itself has no least element since, if x is *any* positive real, then

$$0 < \tfrac{1}{2}x < x.$$

showing that $\frac{1}{2}x$ is a positive real smaller than x. Hence x cannot be the smallest positive real.

From the Well-Ordering Principle we can quickly deduce the result which is the backbone of the method of proof by *Mathematical Induction.*

Theorem 3.1 Principle of Finite Induction

If S is a set of positive integers with the following two properties:

(a) 1 is a member of S;

(b) if $k \in S$, then the next integer $k+1 \in S$;

then $S = \mathbb{Z}^+$.

Proof of Theorem 3.1

We shall give a proof by contradiction. We assume that the theorem is not true and that there is a set S of positive integers satisfying (a) and (b) which is not the whole of $\mathbb{Z}^+$. We show that this assumption leads to a contradiction and therefore the theorem must be true.

Let A be the set of all positive integers which do not belong to S. By the assumption that S is not the whole of $\mathbb{Z}^+$, we know that A is non-empty. Hence, by the Well-Ordering Principle, A must contain a least member. Let this least member of A be a.

By property (a) the integer 1 belongs to S, and so 1 is not in A. Therefore $a > 1$. Now consider the integer $a-1$ which, as $a > 1$, is positive. Furthermore, as $a - 1 < a$ and a is the least member of A, it follows that $a - 1$ does not belong to A. Therefore $a - 1$ is a member of S.

But property (b) now tells us that the integer following $a - 1$, namely a itself, belongs to S. This contradicts the fact that a belongs to A. This contradiction means that the one assumption we made, namely that A is non-empty, must be false. So A is empty and hence S is the set of all positive integers. ■

Example 3.1

Prove that

$$1 + 3 + 5 + \cdots + (2n-1) = n^2, \quad \text{for all positive integers } n.$$

We discovered this formula in Section 2 but did not give a formal proof. We rectify that omission now.

With an eye on the Principle of Finite Induction, let S be the set of positive integers n for which the formula holds. Our goal is to show that $S = \mathbb{Z}^+$.

Putting $n = 1$ in the formula, we observe that the left-hand side has only one term so that the formula reduces to $1 = 1^2$, which is true. So $1 \in S$.

It remains to prove property (b). To that end, suppose $k \in S$, where k is some positive integer. Since $k \in S$ we have that

$$1 + 3 + 5 + \cdots + (2k-1) = k^2.$$

To deduce that $k + 1 \in S$ we must show the truth of the given formula for the case $n = k + 1$. Working on the left-hand side of the formula for $n = k + 1$:

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k-1) + (2k+1) & \\ &= k^2 + (2k+1), \quad \text{from the assumption that } k \in S, \\ &= (k+1)^2, \quad \text{which is the required right-hand side.} \end{aligned}$$

This reasoning shows that if $k \in S$ then $k + 1 \in S$, which confirms property (b).

Hence $S = \mathbb{Z}^+$, and the formula is true for all positive integers. ♦

We presented Example 3.1 by applying the Principle of Finite Induction to the set of integers for which the proposition was true. In future we shall

present such arguments less formally and will refer to such arguments as proof by *Mathematical Induction.* The propositions for which we attempt such proofs are ones which are given in terms of a general integer n and are reportedly true for all $n \in \mathbb{Z}^+$.

Let $P(n)$ be such a proposition and let S be the set of positive integers n for which $P(n)$ is true. If we know that $P(1)$ is true then $1 \in S$. Further, if we can show that whenever $P(k)$ is true it follows that $P(k+1)$ is true, then $k \in S$ implies $k+1 \in S$. Hence, by the Principle of Finite Induction, $S = \mathbb{Z}^+$, and $P(n)$ is true for all positive integers.

The proposition $P(n)$ might take various forms. It might be a formula as in Example 3.1, an inequality such as $2^n > n^3$, which we shall see in Example 3.3, or a statement such as "n can be written as a sum of distinct powers of 2" which we shall meet in Example 3.5.

We state this formally as follows.

> ***Principle of Mathematical Induction***
>
> Let $P(n)$ be a proposition depending on a positive integer n. If:
>
> (a) $P(1)$ is true;
>
> (b) for any integer $k \geq 1$, if $P(k)$ is true then $P(k+1)$ is true;
>
> then $P(n)$ is true for all $n \in \mathbb{Z}^+$.

Step (a), showing that the proposition is true for the first value, is called the *basis for the induction.* Step (b) is called the *induction step.* The assumption made in this step, that $P(k)$ is true for some integer k, is called the *induction hypothesis.*

Example 3.2

Prove the following formula for the sum of the first n triangular numbers.

$$1+3+6+10+\cdots+\tfrac{1}{2}n(n+1) = \tfrac{1}{6}n(n+1)(n+2) \qquad P(n)$$

By 'prove' we mean 'prove for all integers $n \geq 1$'. The 'for all integers $n \geq 1$' is taken for granted.

We use Mathematical Induction to prove that $P(n)$ is true for all integers $n \geq 1$ by establishing (a) and (b) above.

First the basis for the induction. Putting $n = 1$ in $P(n)$ gives

$$1 = \tfrac{1}{6} \times 1(1+1)(1+2).$$

This is correct, showing that $P(1)$ is true.

Now the induction step. We assume that $P(k)$ is true for some positive integer k. That is, we have the induction hypothesis

$$1+3+6+10+\cdots+\tfrac{1}{2}k(k+1) = \tfrac{1}{6}k(k+1)(k+2).$$

We need to deduce the truth of $P(k+1)$ from this, that is

$$1+3+6+10+\cdots+\tfrac{1}{2}k(k+1)+\tfrac{1}{2}(k+1)(k+2) = \tfrac{1}{6}(k+1)(k+2)(k+3).$$

The last term on the left-hand side is the $k+1$ th triangular number.

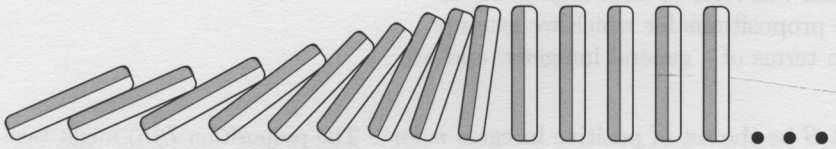
Taking the left-hand side we have

$$\begin{aligned} &1+3+6+10+\cdots+\tfrac{1}{2}k(k+1)+\tfrac{1}{2}(k+1)(k+2) \\ &\quad = \tfrac{1}{6}k(k+1)(k+2)+\tfrac{1}{2}(k+1)(k+2), \quad \text{by the induction hypothesis,} \\ &\quad = (k+1)(k+2)(\tfrac{1}{6}k+\tfrac{1}{2}) \\ &\quad = \tfrac{1}{6}(k+1)(k+2)(k+3), \quad \text{which is the RHS of } P(k+1). \end{aligned}$$

This establishes the truth of $P(k+1)$ and completes the induction step.

Hence, by the Principle of Mathematical Induction, $P(n)$ is true for all positive integers. ♦

Mathematical Induction can be likened to a line of dominoes arranged so that if any falls over it will knock over the next one along the line, (the induction step). Then, if the first domino is knocked over (the basis for the induction), they all fall over.



The first domino is knocked over

If the kth domino is knocked over so too is the $k+1$th

Figure 3.1 Illustrating Mathematical Induction

Of course, the induction step is at the heart of any induction proof. Quite often, as in both the preceding examples, some algebraic manipulation is involved. This may be straightforward or quite complex. Here are a couple for you to try.

Problem 3.2

Use Mathematical Induction to prove that the formula

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1)$$

is true for all positive integers.

Problem 3.3

Use Mathematical Induction to prove the formula for a *geometric series*

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}, \quad r \neq 1.$$

A *geometric series* is one in which the ratio of successive terms is constant, r in this example.

3.3 More general induction

The examples that we have seen using Mathematical Induction have all concerned propositions $P(n)$ which are true for all positive integers. In this case $n = 1$, the smallest integer for which the proposition was true, provided the basis for the induction. In fact there is nothing special about the number 1. The Principle of Finite Induction is readily adapted to provide proofs of results which are true for all integers greater than some integer n_0.

Principle of Mathematical Induction (generalized)

Let $P(n)$ be a proposition depending on an integer n. If:

(a) $P(n_0)$ is true;

(b) for any integer $k \geq n_0$, if $P(k)$ is true then $P(k+1)$ is true;

then $P(n)$ is true for all integers $n \geq n_0$.

As before, step (a) is called the basis for the induction, and step (b) is called the induction step.

In the next example we shall see such an induction proof when we prove a result that is true for all integers from 10 onwards. In this example the induction step involves more reasoning than the simple algebraic manipulations that we have met so far.

Example 3.3

For which positive integers n is it true that $2^n > n^3$?

Exploration of the relative values of 2^n and n^3 for small values of n shows that 2^n is larger when $n = 1$, but then n^3 is larger for $n = 2, 3, 4, \ldots, 9$. For $n = 10$, 2^n becomes the larger once again:

$$2^{10} = 1024 > 1000 = 10^3.$$

As n now increases it appears that 2^n grows more quickly than n^3, which leads us to conjecture:

$$2^n > n^3 \text{ for all integers } n \geq 10.$$

Preparing the way for induction, we let $P(n)$ be the statement that $2^n > n^3$.

Having seen that $P(10)$ is true, the basis for the induction is established (as $n_0 = 10$).

It remains to show that, for any integer $k \geq 10$, if $P(k)$ is true then $P(k+1)$ is true. So the induction hypothesis is

$$2^k > k^3, \quad \text{where } k \geq 10,$$

and from this assumption we require to show that

$$2^{k+1} > (k+1)^3.$$

Now $2^{k+1} = 2 \times 2^k$ and so the induction hypothesis gives

$$2^{k+1} > 2k^3.$$

Therefore it suffices to show that, for any $k \geq 10$,

$$2k^3 \geq (k+1)^3,$$

or rearranging,

$$\left(1 + \frac{1}{k}\right)^3 \leq 2.$$

This is certainly true by the following argument. As $k > 1$ the largest value of $\left(1 + \frac{1}{k}\right)^3$ comes from the largest value of $1 + \frac{1}{k}$. This in turn comes from the smallest value of k, namely $k = 10$. However, $\left(1 + \frac{1}{10}\right)^3 = 1.331$, which is less than 2. This completes the induction step.

Hence, by the Generalized Principle of Mathematical Induction, the proposition is true for all integers $n \geq 10$. ♦

Example 3.3 has some points of interest. Investigation of the induction step led us to the inequality

$$\left(1 + \frac{1}{k}\right)^3 \leq 2.$$

In fact this inequality holds true for all integers $k \geq 4$ and so the induction step does hold for all integers $k \geq 4$, (although we were only concerned with integers $k \geq 10$). Drawing on the earlier analogy with dominoes the situation in Example 3.3 is as depicted in Figure 3.2. If any domino from the fourth onwards were to fall it would knock the next over, but the first one of these which *does* fall is the tenth.

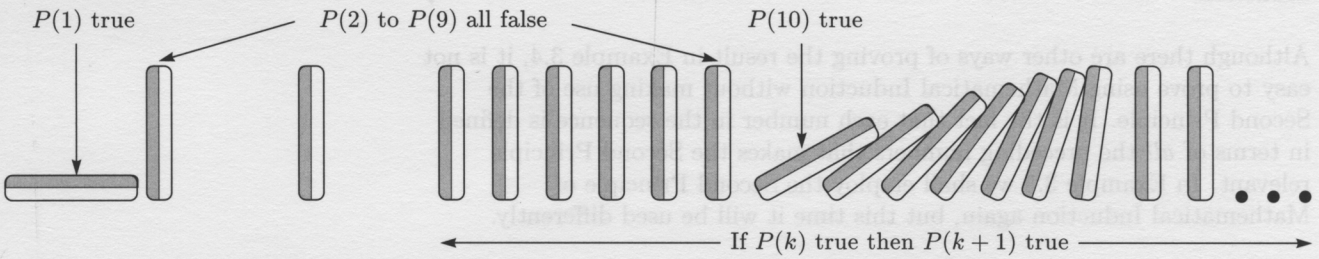


Figure 3.2 $P(n) : 2^n > n^3$ *for* $n \geq 10$

We shall have occasion to use another variant of Mathematical Induction. The *Second Principle of Mathematical Induction* employs a different induction step.

> ***Second Principle of Mathematical Induction***
>
> Let $P(n)$ be a proposition depending on an integer n. If:
>
> (a) $P(n_0)$ is true;
>
> (b′) for any integer $k \geq n_0$, if $P(n_0)$, $P(n_0+1)$, $\ldots$, $P(k)$ are all true, then $P(k+1)$ is true;
>
> then $P(n)$ is true for all integers $n \geq n_0$.

The Second Principle of Mathematical Induction seems logically sound if one thinks of the domino analogy, and can be established from the Well-Ordering Principle. We shall not do so here but proof of this can be found in the solution to Exercise 4 of Section 3 in the Additional Exercises.

The next two examples both make use of the Second Principle of Mathematical Induction.

Example 3.4

A sequence of integers is defined as follows:

$$x_0 = 1; \quad x_n = x_0 + x_1 + \cdots + x_{n-1}, \quad \text{for all integers } n \geq 1.$$

Prove that $x_n = 2^{n-1}$, for all integers $n \geq 1$.

Denote the proposition $x_n = 2^{n-1}$ by $P(n)$.

Then $x_1 = x_0 = 1 = 2^0$, so $P(1)$ is true and we have the basis for the induction.

Heading towards the (new) induction step, suppose that $P(r)$ is true for all $1 \leq r \leq k$; that is,

$$x_r = 2^{r-1}, \quad \text{for } 1 \leq r \leq k.$$

Then

$$\begin{aligned} x_{k+1} &= x_0 + x_1 + \cdots + x_k \\ &= 1 + \left(1 + 2 + 4 + \cdots + 2^{k-1}\right), \quad \text{by the induction hypothesis,} \\ &= 1 + \frac{2^k - 1}{2 - 1}, \quad \text{by the formula for a geometric series, Problem 3.3,} \\ &= 2^k. \end{aligned}$$

This establishes the truth of $P(k+1)$ and completes the induction step.

Hence the result follows from the Second Principle of Mathematical Induction. ♦

Although there are other ways of proving the result in Example 3.4, it is not easy to prove using Mathematical Induction without making use of the Second Principle. It is the fact that each number in the sequence is defined in terms of *all* the preceding numbers that makes the Second Principle relevant. In Example 3.5 we shall employ the Second Principle of Mathematical Induction again, but this time it will be used differently.

Example 3.5

Prove that every positive integer n can be written as a sum of distinct (integer) powers of 2. For example, $13 = 2^0 + 2^2 + 2^3$ and $20 = 2^2 + 2^4$. In fact each positive integer is expressed *uniquely* in this way, but we shall leave the uniqueness aside here.

Let $P(n)$ be the proposition that n can be written as a sum of distinct powers of 2.

Then, since $1 = 2^0$, $P(1)$ is true and we have the basis for the induction.

To make use of the Second Principle we assume that, for some positive integer k, $P(r)$ is true for each integer r in the range $1 \leq r \leq k$. That is, we assume that each r in this range can be expressed in the required way. To complete the induction proof we show that, under this assumption, $k+1$ can be expressed as a sum of distinct powers of 2, thereby establishing the truth of $P(k+1)$.

We consider two cases, depending on whether $k+1$ is even or odd.

$k+1$ even

Since $k+1$ is even we can write $k+1 = 2s$ for some positive integer s. Now $s < k+1$ so, by the induction hypothesis, s can be written as a sum of distinct powers of 2, say

$$s = 2^a + 2^b + \cdots + 2^f, \quad a, b, \ldots, f \text{ distinct.}$$

Multiplying this sum by 2 gives

$$k+1 = 2s = 2^{a+1} + 2^{b+1} + \cdots + 2^{f+1}, \quad a+1, b+1, \ldots, f+1 \text{ distinct,}$$

as required.

$k+1$ odd

Since $k+1$ is odd we can write $k+1 = 2s+1$ for some positive integer s. Now $2s < k+1$ so, by the induction hypothesis, $2s$ can be written as a sum of distinct powers of 2, say

$$2s = 2^a + 2^b + \cdots + 2^f, \quad a, b, \ldots, f \text{ distinct.}$$

Since $2s$ is even, and the exponents in this representation of $2s$ are distinct, no exponent can be 0.

The only power of 2 giving an odd integer is $2^0 = 1$.

Hence

$$\begin{aligned} k+1 &= 2s+1 \\ &= 2^a + 2^b + \cdots + 2^f + 1, \quad a, b, \ldots, f \text{ distinct,} \\ &= 2^a + 2^b + \cdots + 2^f + 2^0, \quad a, b, \ldots, f, 0 \text{ distinct,} \end{aligned}$$

as required.

This completes the induction step and the result follows by the Second Principle of Mathematical Induction. ♦

Problem 3.4

In the sequence

$$1,\ 3,\ 4,\ 7,\ 11,\ 18,\ 29,\ 47,\ \ldots,$$

each term, from the third onwards, is the sum of the previous two terms. That is, the sequence $\{L_n\}$ is defined by

$$L_1 = 1; \quad L_2 = 3; \quad L_n = L_{n-1} + L_{n-2}, \quad \text{for } n \geq 3.$$

Use the Second Principle of Mathematical Induction to prove that

$$L_n < \left(\tfrac{7}{4}\right)^n, \quad \text{for all } n \geq 1.$$

Mathematical Induction provides a powerful method of proof, whose applications include proving the truth of formulae for all integers from some integer onwards. But while Mathematical Induction is a great help in proving formulae, it does not help us to find such formulae. Discovering the formulae in the first place is a separate activity.

4 DIVISIBILITY

4.1 Integer division

From an early age we learn how to divide one positive integer by another obtaining a quotient and a remainder. For example, we can divide 23 by 4 to obtain a quotient of 5 and a remainder of 3 or, rather more formally, $23 = 4 \times 5 + 3$. In fact if we stipulate that when dividing by 4 the only permitted remainders are 0, 1, 2, and 3 then the quotient and remainder in any division by 4 turn out to be unique. That is, the only way that we can write $23 = 4 \times q + r$, where q and r are integers with $0 \leq r < 4$, is $q = 5$ and $r = 3$. Such uniqueness holds for division by any positive integer. The familiar result exemplified here, known as the *Division Algorithm*, is of enormous theoretical importance and in what follows we shall develop certain consequences of it which are fundamental to our treatment of numbers.

Theorem 4.1 The Division Algorithm

For any two integers a and b, where $b > 0$, there exist unique integers q and r such that

$$a = bq + r, \quad \text{where } 0 \leq r < b.$$

Proof of Theorem 4.1

Consider the set of integers

$$S = \{a - bn \geq 0 : n \in \mathbb{Z}\}.$$

S is the set of non-negative integers in the set $\{\ldots,\ a - 2b,\ a - b,\ a,\ a + b,\ \ldots\}$.

The set S is non-empty because, as b is positive, $a - bn$ will certainly be positive for large negative values of n. Either 0 is a member of S, and hence the least member of S, or S is a non-empty set of positive integers and, by the Well-Ordering Principle, again S has a least member. So in either case there exists an integer q such that $a - bq = r$ is the least member of S. Hence we have integers q and r such that

$$a = bq + r, \quad \text{where } 0 \leq r.$$

If $r \geq b$ then $r - b \geq 0$. In addition

$$r - b = a - bq - b = a - b(q + 1)$$

and so $r - b$ is a member of S. However $r - b$ is smaller than r, which contradicts the minimality of r, and so $r < b$.

It remains to prove the uniqueness of q and r. Suppose, therefore, that

$$a = bq + r, \quad 0 \leq r < b,$$

and

$$a = bq' + r', \quad 0 \leq r' < b.$$

Subtracting gives

$$0 = b(q - q') + (r - r').$$

Now if $q = q'$ this equation gives $r = r'$ and so the expressions for a are the same.

On the other hand, if $q \neq q'$, we may assume that $q > q'$ and hence that $q - q' > 0$. It follows that $q - q' \geq 1$. Since $b > 0$ we have

$$\begin{aligned} r' - r &= b(q - q') \\ &\geq b \times 1 = b. \end{aligned}$$

Adding r to both sides gives

$$r' \geq b + r \geq b,$$

contradicting the definition of r'.

Therefore the values of q and r are unique. ■

Essentially the same argument is illustrated as follows. Imagine multiples of b stepped off along the number line, as shown in Figure 4.1. As the intervals marked off in this way cover the whole line, the number a lies in some interval

$$bq \leq a < b(q+1).$$

This interval containing a will be unique. Notice that if a coincides with one of the multiples of b the inequalities decree that we choose the interval which has a at its left-hand end.

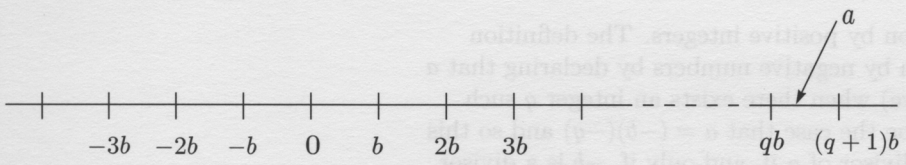


Figure 4.1 Illustrating the Division Algorithm

Subtracting bq through the above inequality, shows us that there is a unique integer q such that $0 \leq a - bq < b$. Now putting $a - bq = r$ gives the claimed result.

The way of representing numbers suggested by the Division Algorithm will be important to us, so let us pause to look at some simple applications.

Example 4.1

Show that when any square is divided by 4 the remainder is either 0 or 1.

The Division Algorithm, applied to division by 4, tells us that any integer can be written uniquely in one of the forms $4n$, $4n+1$, $4n+2$ or $4n+3$ for some integer n. So any square number is the square of an integer of one of these four forms. We write each square in its unique form $4q + r$ and hope to find that the only possible values for r are 0 and 1.

$$(4n)^2 = 16n^2 = 4(4n^2) + 0$$
$$(4n+1)^2 = 16n^2 + 8n + 1 = 4(4n^2 + 2n) + 1$$
$$(4n+2)^2 = 16n^2 + 16n + 4 = 4(4n^2 + 4n + 1) + 0$$
$$(4n+3)^2 = 16n^2 + 24n + 9 = 4(4n^2 + 6n + 2) + 1$$

In each case the remainder on division by 4 is 0 or 1, as claimed.

In fact we could have made the calculations simpler by working with remainders on division by 2. Any number is either of form $2n$ or of form $2n+1$ and the respective squares are

$$(2n)^2 = 4n^2 + 0,$$
$$(2n+1)^2 = 4n^2 + 4n + 1 = 4(n^2 + n) + 1,$$

showing that the remainder on division by 4 is either 0 or 1. ♦

Problem 4.1

The Division Algorithm tells us that any integer can be written in one of the forms $3n$, $3n+1$ or $3n+2$. Use this fact to deduce that when any cube is divided by 9 the remainder is one of 0, 1 or 8.

The Division Algorithm readily provides us with the needed definition of divisibility; one number is divisible by another when the unique remainder is 0.

Definition 4.1 Divisors and multiples

An integer a is divisible by the positive integer b, (or, for brevity, b divides a), if there exists some integer q such that $a = bq$.

When b divides a we say that b is a *divisor* of a or that a is a *multiple* of b.

We write $b \mid a$ as a shorthand for b divides a, and $b \nmid a$ for b does not divide a.

In other texts you may find the phrase 'b is a *factor* of a' rather than 'b is a divisor of a'.

So, for example, 21 is divisible by 3 since $21 = 3 \times 7$ and $6 \mid -24$ since $-24 = 6 \times (-4)$. On the other hand $6 \nmid 16$ since $16 = 6 \times 2 + 4$.

Notice that we only consider division by positive integers. The definition could be adapted to permit division by negative numbers by declaring that a is divisible by b (positive or negative) when there exists an integer q such that $a = bq$. But if $a = bq$ it must be the case that $a = (-b)(-q)$ and so this definition would lead to b being a divisor of a if, and only if, $-b$ is a divisor of a. There is therefore nothing essentially complicated in dividing by negative numbers, but we shall have no occasion to do so. Therefore, if we ask for the divisors of an integer, we shall be expecting positive answers.

Notice also that 0 is excluded as a divisor. However, since $0 = n \times 0$ it is a multiple of every positive integer, and 0 is unique in having an infinite number of divisors as we shall see.

Problem 4.2

List all the divisors of 18 and -24. Do these two numbers have any divisors in common?

Problem 4.3

Show that if n is not a multiple of 3 then $n^2 - 1$ is divisible by 3.

Problem 4.2 highlights one property of divisibility, namely that any integer greater than 1 has at least two divisors, namely itself and 1. This is just one of a number of simple properties that are direct consequences of the definition, properties which are not particularly exciting in themselves but which will prove to be useful as refined tools for deriving further information. A selection of the more important properties have been grouped together in the following theorem.

Theorem 4.2 Properties of division

Let a and b be positive integers and c and d be any integers.

(a) If $a \mid c$ then $a \mid (c + na)$ for any integer n.

(b) If $c \neq 0$ and $a \mid c$ then $a \leq |c|$.

(c) If $a \mid b$ and $b \mid a$ then $a = b$.

(d) If $a \mid b$ and $b \mid c$ then $a \mid c$.

(e) If $a \mid c$ and $a \mid d$ then $a \mid (mc + nd)$ for any integers m and n.

Proof of Theorem 4.2

(a) As $a \mid c$ there is an integer q such that $c = aq$. But then

$$c + na = aq + na = a(q + n) = ax,$$

and as $x = q + n$ is an integer, this confirms that a divides $c + na$.

(b) As $c = aq$ as in (a), $|c| = |aq| = a|q|$, and the result follows since $|q| \geq 1$, q being a non-zero integer.

Notice that this result shows that the non-zero integer c can have only a finite number of divisors, since they must all be less than or equal to $|c|$.

(c) If $a \mid b$ and $b \mid a$ then, from (b), $a \leq b$ and $b \leq a$. The required equality follows.

(d) If $a \mid b$ and $b \mid c$ then there are integers s and t such that $b = as$ and $c = bt$. Substituting for b gives $c = ast = a(st)$, and as st is an integer this shows that $a \mid c$.

(e) If $a \mid c$ and $a \mid d$ then there are integers s and q such that $c = as$ and $d = aq$. But then, for any integers m and n,

$$mc + nd = mas + naq = a(ms + nq) = ax,$$

where $x = ms + nq$ is an integer. So $a \mid (mc + nd)$. ■

4.2 The greatest common divisor

In Problem 4.2 we saw that the two numbers 18 and -24 had four common divisors, namely 1, 2, 3 and 6. It is clear that any two integers will have at least one common divisor because 1 divides every integer. What is more, property (b) of the previous theorem guarantees that any integer (other than 0) has only finitely many divisors. Therefore any two integers, not both of which are 0, have a non-empty, finite set of common divisors. We are going to be interested in the *greatest common divisor* of two integers, that is the largest member of the finite set of common divisors.

Definition 4.2 Greatest common divisor

The greatest common divisor of two integers a and b, not both of which are zero, is the positive integer n satisfying:

(a) $n \mid a$ and $n \mid b$;

(b) if $d \mid a$ and $d \mid b$ then $d \leq n$.

The greatest common divisor of a and b is denoted by $\gcd(a, b)$.

Greatest common divisors are also known as highest common factors with the corresponding notation $\mathrm{hcf}(a, b)$.

When we write $\gcd(a, b)$ it will be understood that a and b are integers not both zero.

Property (a) says that $\gcd(a, b)$ *is* a common divisor of a and b while property (b) says that it is *maximal* amongst the common divisors.

Problem 4.2 establishes that $\gcd(18, -24) = 6$. Similarly, as you may wish to verify, $\gcd(3, 20) = 1$ and $\gcd(14, 35) = 7$.

Two properties which follow immediately from the definition are worth recording.

Since zero is divisible by every positive integer, it follows that

$$\gcd(0, b) = |b|.$$

Since n and $-n$ have the same set of positive divisors,

$$\gcd(a, b) = \gcd(-a, b) = \gcd(a, -b) = \gcd(-a, -b).$$

Property (e) of Theorem 4.2 shows that any common divisor of two integers c and d also divides any integer of the form $mc + nd$. Numbers of this latter form merit their own name.

Definition 4.3 Integer combination

If a and b are integers, then any integer of the form

$$ma + nb, \quad m, n \in \mathbb{Z},$$

is called an *integer combination* of a and b.

So $\gcd(a, b)$ divides every integer combination of a and b. In fact more is true: one of these integer combinations *is* $\gcd(a, b)$.

To illustrate that claim, take the three examples of greatest common divisors which we have seen.

$$\gcd(18, -24) = 6 = 3 \times 18 \ + \ 2 \times (-24)$$
$$\gcd(3, 20) = 1 = 7 \times 3 \ + \ (-1) \times 20$$
$$\gcd(14, 35) = 7 = (-2) \times 14 \ + \ 1 \times 35$$

Theorem 4.3

Given any integers a and b, not both zero, there exist integers m and n such that

$$\gcd(a, b) = ma + nb.$$

In other words, $\gcd(a, b)$ is an integer combination of a and b.

Proof of Theorem 4.3

Consider the set

$$S = \{xa + yb > 0 : x, y \text{ integers}\}$$

of all positive integer combinations of a and b. Each member of S is an integer and S is non-empty because a, $-a$, b and $-b$ are all integer combinations of a and b at least one of which is positive. Hence the Well-Ordering Principle guarantees the existence of a smallest element d in S. As d is a member of S we know that $d = ma + nb$ for some integers m and n. It is our contention that $d = \gcd(a, b)$.

We first show that d is a common divisor of a and b. The Division Algorithm tells us that, from division of a by d, we have $a = dq + r$ for some integers q and r with $0 \leq r < d$. But then

$$\begin{aligned} r = a - dq &= a - (ma + nb)q \\ &= (1 - mq)a + (-nq)b, \end{aligned}$$

showing that r is an integer combination of a and b. Now if it were the case that $r > 0$ then r would be a member of S and the condition $r < d$ would contradict d being the least member of S. Hence $r = 0$ and d is a divisor of a. Similarly d is also a divisor of b.

It remains to show that d is the *greatest* of the common divisors. Suppose then that d' is any common divisor of a and b. Then, by property (e) of Theorem 4.2, d' is a divisor of the integer combination $ma + nb$; that is, d' is a divisor of d. Property (b) of Theorem 4.2 then gives $d' \leq d$, as required. ■

The ability to write the greatest common divisor of two integers as an integer combination of them will prove to be very useful. Unfortunately, though our theorem guarantees that an integer combination exists, it gives us no clue as to how to find that integer combination. We shall return to that problem shortly, but first we shall establish a number of useful consequences of the theorem. The first two concern pairs of integers for which the greatest common divisor is 1. We have a term to describe such pairs.

Definition 4.4 Relatively prime

Two integers a and b, not both zero, are said to be *relatively prime* whenever $\gcd(a, b) = 1$.

Some authors use the term 'coprime' as an alternative to 'relatively prime'.

For example, 10 and 21 are relatively prime, since 10 and 21 do not have any common divisors other than 1.

We exclude the case where a and b are both zero because $\gcd(0, 0)$ does not exist. For any non-zero integer a, $\gcd(0, a) = |a|$ and hence the only integers which are relatively prime to 0 are 1 and -1.

Theorem 4.4

Integers a and b are relatively prime if, and only if, there exist integers m and n such that $1 = ma + nb$.

Proof of Theorem 4.4

If a and b are relatively prime then $\gcd(a, b) = 1$ and Theorem 4.3 confirms the existence of integers m and n with $ma + nb = 1$.

Conversely, suppose there are integers m and n with $ma + nb = 1$, and let d be any common divisor of a and b. Then property (e) of Theorem 4.2 tells us that d divides $ma + nb$; that is, d divides 1. Finally property (b) of Theorem 4.2 gives $d = 1$, and $\gcd(a, b) = 1$, as claimed. ■

Notice that the integers m and n in the statement of Theorem 4.4 are also relatively prime. Indeed, if there exist non-zero integers a, b, m and n such that $ma + nb = 1$ then each of a and m is relatively prime to each of b and n.

Theorem 4.5

For any integers a and b, not both zero, if $\gcd(a, b) = d$ then $\frac{a}{d}$ and $\frac{b}{d}$ are integers such that $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

That is, the integers $\dfrac{a}{\gcd(a, b)}$ and $\dfrac{b}{\gcd(a, b)}$ are relatively prime.

We shall often use this in the following form.

If $\gcd(a, b) = d$ there exist integers s and t such that $a = ds$ and $b = dt$, with $\gcd(s, t) = 1$.

The s and t are respectively $\frac{a}{d}$ and $\frac{b}{d}$.

Proof of Theorem 4.5

Since d divides a and d divides b there exist integers s and t such that $a = ds$ and $b = dt$. Moreover Theorem 4.3 gives the existence of integers m and n such that $d = ma + nb$. Dividing this equation through by d leaves us with $1 = ms + nt$, whereupon Theorem 4.4 informs us that $\gcd(s, t) = 1$. That is,

$$\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1. \quad ■$$

In the statement of the theorem, we used $\frac{a}{d}$ and $\frac{b}{d}$, knowing that they were integers. In general it is advisable to avoid the use of fractions when trying to prove results concerning integers. Note how, in the proof, we introduced the integers s and t instead of using $\frac{a}{d}$ and $\frac{b}{d}$ respectively.

Diversion

Throughout the course units we shall, occasionally, break the flow by displaying some curious pattern of behaviour or presenting some interesting fact about numbers. We call these interruptions 'diversions'. Make of them what you will.

$$\begin{aligned} 11 &= 1 \times 9 + 2 \\ 111 &= 12 \times 9 + 3 \\ 1111 &= 123 \times 9 + 4 \\ 11111 &= 1234 \times 9 + 5 \\ 111111 &= 12345 \times 9 + 6 \\ 1111111 &= 123456 \times 9 + 7 \\ 11111111 &= 1234567 \times 9 + 8 \\ 111111111 &= 12345678 \times 9 + 9 \\ 1111111111 &= 123456789 \times 9 + 10 \end{aligned}$$

We shall take a break from the string of results at this point and let you try your hand at some simple proofs. Each of the exercises in Problem 4.4 involves little more than an appropriate application of one of the preceding theorems.

Problem 4.4

Let a and b be integers, not both of which are zero. Prove the following.

(a) The integer c is a multiple of $\gcd(a, b)$ if, and only if, there exist integers x and y such that $c = xa + yb$.

(b) If $\gcd(a, b) = ma + nb$ then $\gcd(m, n) = 1$.

(c) If $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$ then $\gcd(a, bc) = 1$.

(d) For any positive integer k, $\gcd(ka, kb) = k \times \gcd(a, b)$.

Note that the result in part (a) asserts that the set of all integer combinations of a and b is precisely the set of all integer multiples of $\gcd(a, b)$.

The next two results establish further divisibility properties.

Theorem 4.6

If $a \mid c$ and $b \mid c$, with $\gcd(a, b) = 1$, then $ab \mid c$.

Proof of Theorem 4.6

As a divides c and b divides c there are integers r and s such that

$$c = ar \quad \text{and} \quad c = bs.$$

Also, $\gcd(a, b) = 1$ implies the existence of integers m and n such that

$$ma + nb = 1.$$

Multiplying through this equation by c, we obtain

$$cma + cnb = c.$$

Substituting bs for the first c and ar for the second c in the left-hand side of this equation gives $ab(sm + rn) = c$, which confirms that ab divides c. ■

Theorem 4.7 Euclid's Lemma

If $a \mid bc$, with $\gcd(a, b) = 1$, then $a \mid c$.

Proof of Theorem 4.7

From a divides bc and $\gcd(a, b) = 1$ we have integers k, m and n such that

$$bc = ka \quad \text{and} \quad ma + nb = 1.$$

Multiplying through the latter by c gives

$$mac + nbc = c$$

and then substitution for bc leaves us with

$$mac + nka = c.$$

That is, $a(mc + nk) = c$, which confirms that a divides c, as required. ■

The following examples incorporate several of the preceding ideas.

Example 4.2

Suppose that a divides bc, where $\gcd(b, c) = 1$. Show that a can be written as a product rs, where r divides b and s divides c.

To illustrate the result, notice that 21 divides 6×35, and $21 = 3 \times 7$, where 3 divides 6 and 7 divides 35, and this is the only way of expressing 21 as a product of two numbers to fit the statement.

Let $r = \gcd(a, b)$. By Theorem 4.5, there exist integers s and t such that

$$a = rs \quad \text{and} \quad b = rt, \quad \text{where } \gcd(s, t) = 1.$$

Hence we have written $a = rs$, where r divides b. It remains to show that s divides c.

Since a divides bc there exists an integer k such that

$$bc = ka.$$

That is, substituting for a and b,

$$rtc = krs.$$

Cancelling the r on both sides gives

$$tc = ks,$$

which implies that s divides tc. However, s and t are relatively prime and so, by Euclid's Lemma, s divides c, as required. ♦

In fact the only integers r and s to fit the statement are $r = \gcd(a, b)$ and, by symmetry, $s = \gcd(a, c)$.

Example 4.3

Show that if a is an integer with $\gcd(6, a) = 1$ then 24 divides $a^2 - 1$.

We shall show that 3 divides $a^2 - 1$ and 8 divides $a^2 - 1$. The result then follows from Theorem 4.6, since $\gcd(3, 8) = 1$.

Thinking about division by 3, we note that a must be of one of the forms $3n$, $3n + 1$ or $3n + 2$. However, since $\gcd(6, 3n) \geq 3$, the condition that $\gcd(6, a) = 1$ rules out the first of these possibilities. For the case $a = 3n + 1$ we have

$$a^2 - 1 = (3n + 1)^2 - 1 = 9n^2 + 6n + 1 - 1 = 3(3n^2 + 2n),$$

and for the case $a = 3n + 2$ we have

$$a^2 - 1 = (3n + 2)^2 - 1 = 9n^2 + 12n + 4 - 1 = 3(3n^2 + 4n + 1).$$

So in each case $a^2 - 1$ is divisible by 3.

Turning to division by 8, there are the eight possible forms for a, namely $8n + r$, for $r = 0, 1, \ldots, 7$. However we notice that $\gcd(a, 6) = 1$ tells us

that a cannot be even. Thus we are left with the odd cases which are $8n+1$, $8n+3$, $8n+5$ and $8n+7$. Taking these four cases in turn we have:

$$(8n+1)^2 - 1 = 64n^2 + 16n + 1 - 1 = 8(8n^2 + 2n);$$
$$(8n+3)^2 - 1 = 64n^2 + 48n + 9 - 1 = 8(8n^2 + 6n + 1);$$
$$(8n+5)^2 - 1 = 64n^2 + 80n + 25 - 1 = 8(8n^2 + 10n + 3);$$
$$(8n+7)^2 - 1 = 64n^2 + 112n + 49 - 1 = 8(8n^2 + 14n + 6).$$

In each case $a^2 - 1$ is divisible by 8.

This completes the proof. ♦

And now some for you to try.

Problem 4.5

Suppose that $\gcd(a, b) = 1$. Use Theorem 4.4 to prove the following.

(a) If $c \mid a$ then $\gcd(b, c) = 1$.

(b) $\gcd(a, a + b) = 1$.

Problem 4.6

Prove the following statements are true for any integer n.

(a) $\gcd(3n + 2, 5n + 3) = 1$. *Hint*: Use property (e) of Theorem 4.2 in the following way. Suppose that d is any common divisor of $3n + 2$ and $5n + 3$. Then d divides $(r(3n + 2) + s(5n + 3))$ for any integers r and s. Try to choose r and s so that d divides 1.

(b) $\gcd(n^2 + 3n + 1, 2n + 1) = 1$. *Hint*: This time you may have to apply property (e) of Theorem 4.2 more than once.

4.3 The least common multiple

In contrast to common divisors we also have the notion of common multiples. For example, since $6 \mid 60$ and $10 \mid 60$ we say that 60 is a common multiple of 6 and 10. Of course there are infinitely many common multiples of 6 and 10, the positive ones including 30, 60, 90, 120, 150, 180, and so on. Indeed it would appear that the common multiples of 6 and 10 are precisely the multiples of 30, 30 being the *least common multiple* of 6 and 10.

Definition 4.5 Least common multiple

The least common multiple of the non-zero integers a and b is the positive integer n satisfying:

(a) $a \mid n$ and $b \mid n$;

(b) if $a \mid m$ and $b \mid m$ then $n \leq |m|$.

The least common multiple of a and b is denoted by $\text{lcm}(a, b)$.

Property (a) of this definition says that $\text{lcm}(a, b)$ is a common multiple of a and b while property (b) says that it is the smallest positive integer with this property. The fact that $|ab|$ is a common multiple of a and b, coupled with the Well-Ordering Principle, ensures that $\text{lcm}(a, b)$ exists.

The example preceding the definition suggested something stronger than (b), namely that $\text{lcm}(a, b)$ divides any common multiple of a and b. As a first exercise in handling common multiples we invite you to prove that this property does indeed follow from the definition.

Problem 4.7

Show that if m is a common multiple of the non-zero integers a and b then $\text{lcm}(a, b)$ divides m. *Hint*: Use the Division Algorithm and show that the remainder must be zero.

Not surprisingly the twin concepts of greatest common divisor and least common multiple are interrelated. Recall from the earlier numerical example that $\text{lcm}(6, 10) = 30$ and $\gcd(6, 10) = 2$. It is no accident that $\text{lcm}(6, 10) \times \gcd(6, 10) = 6 \times 10$.

Theorem 4.8

For any pair of positive integers a and b,

$$\text{lcm}(a, b) \times \gcd(a, b) = ab.$$

The corresponding result in which either a or b is negative, is

$$\text{lcm}(a, b) \times \gcd(a, b) = |ab|.$$

It is readily deduced from this one.

Proof of Theorem 4.8

Let $d = \gcd(a, b)$. Then since d divides a, it also divides ab. So there exists a positive integer n such that

$$ab = dn.$$

Our task is to show that $n = \text{lcm}(a, b)$.

Since d divides both a and b, there exist integers u and v such that

$$a = du \quad \text{and} \quad b = dv.$$

Substituting for a and then b in $ab = dn$, and cancelling a d in each case gives

$$n = bu \quad \text{and} \quad n = av.$$

This shows that n is a common multiple of a and b, which leaves us to show it is the least such.

Let m be any other common multiple of a and b. There exist integers r and s such that

$$m = ar \quad \text{and} \quad m = bs.$$

We also know that $d = \gcd(a, b)$ is an integer combination of a and b, say

$$d = ax + by.$$

Multiplying this equation by m, and substituting first bs and then ar for m on the right-hand side, we get the following.

$$\begin{aligned} md &= max + mby \\ &= bsax + arby \\ &= ab(sx + ry) \\ &= dn(sx + ry), \quad \text{from the third line of the proof.} \end{aligned}$$

Lastly, cancellation of the d gives

$$m = n(sx + ry).$$

This shows that n is a divisor of m and so $n \leq |m|$ as required. ■

Problem 4.8

Use Theorem 4.8 to determine:

(a) $\text{lcm}(18, 24)$; (b) $\text{lcm}(27, 45)$; (c) $\text{lcm}(39, 40)$.

5 THE LINEAR DIOPHANTINE EQUATION

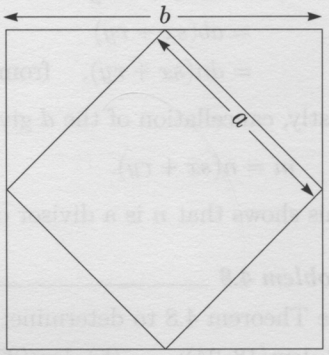
5.1 Pythagoras and Euclid

Although the origin of numbers can be traced back a long way in history, the first serious study of numbers as a theory is attributed to the early Greeks and, in particular, to Pythagoras and his disciples. Little is known with any certainty about the life of Pythagoras. He was born around 569 BC on the Aegean island of Samos. He appears to have studied in both Egypt and Asia Minor before migrating with his mother to Sicily. From there he opened his own school in the Greek colony of Croton in what is now South East Italy. His schools were a great success, drawing enthusiastic audiences. From the schools he established a secret brotherhood which became known as the *Pythagoreans*. It is said that Pythagoras gave two categories of lecture: one for the Pythagoreans alone and the other for general audiences.

The Pythagoreans adhered to strict codes of conduct, including vegetarianism. But perhaps most significant was the confidentiality maintained in the pursuit of mathematical studies; the discoveries of the Pythagoreans were to be kept within the brotherhood. According to legend one Pythagorean was drowned for violating his oath and boasting in public that he had added the dodecahedron to the list of regular solids known to Pythagoras. This strict discipline and secret organization led to their domination of local politics in Croton, but growing unpopularity culminated in a local revolt in around 500 BC during which Pythagoras himself, and many other members of the organization, were killed. Although their political influence was now destroyed, the Pythagoreans quickly reformed as a mathematical and philosophical society, and they prospered for the best part of two centuries. The secrecy that had been practised in Pythagoras' day was gradually phased out and treatises containing much of the Pythagoreans' teachings and doctrines were published.

Pythagoras divided mathematics into four classes: arithmetic, music, geometry and astronomy. It is the contributions to geometry for which Pythagoras will be best remembered. Pythagoras collated, and proved, a number of results from plane geometry, particularly ones involving triangles and circles. Amongst these was the famous result on right-angled triangles which bears his name. In arithmetic, Pythagoras was mainly concerned with problems in number theory of four sorts: numbers from patterns, ratio and proportion, divisors of numbers and numbers in series. Pythagoras used geometric ideas to interpret and treat many of these problems.

The Pythagoreans were absorbed in number mysticism. Their motto is said to have been 'All is number'. This reflects Pythagoras' own preachings that everything, physical, metaphysical, mathematical, is built on the positive integers 1, 2, 3, ... The number 1, they argued, is the number of reason from which all other numbers are generated. The number 2 is the first female, or even, number and 3 is the first male, or odd, number. The number 5 $(= 2 + 3)$ represented marriage, being the union of the first male and first female, while 8 held the secret of love since it added potency (3) to the marriage (5). Each number had its own attributes and the Greeks invariably had a name for it. But what is arguably Pythagoras' greatest discovery in number theory devastated this whole philosophy. He proved that it is impossible to find two whole numbers such that the square of one of them is equal to twice the square of the other. And yet squares in which one has area equal to twice the other are readily constructed. From this he deduced the existence of irrational numbers.



$b^2 = 2a^2$

Following Pythagoras, the development of number theory moved to Alexandria where, in around 300 BC, the first 'university' was founded. It

had spacious lecture rooms, libraries, museums and laboratories. For almost a thousand years, until it declined under the early Christians and finally was sacked by the Arabs in 641 AD, Alexandria stood as the centre for mathematical development. The founder of the Mathematics school at Alexandria was Euclid.

Very little is known about Euclid other than his being of Greek origin, born around 330 BC. Indeed, the name Euclid has become synonymous with the books that he wrote rather than with the man himself. Of several books, his reputation rests on his one best-seller, *Elements*. Over one thousand editions of this masterpiece have appeared since the first printed version in 1482 and, to this day, very few books have been circulated and studied more widely.

Euclid is mainly remembered as a geometer. However, the *Elements* is a compilation of all mathematical knowledge at the time of Euclid, and it contains much more than geometry. *Elements* is divided into 13 books (or chapters) of which the first six present a treatise on elementary plane geometry, book X is on incommensurables and the final books, XI, XII and XIII, turn to the topic of solid geometry. The three that we passed over, books VII, VIII and IX, are on number theory. Book VII is concerned with various types of number, odd, even, square, prime, composite etc., and includes 'Euclid's Algorithm', which we are about to meet. Book VIII is concerned with simple properties of squares and cubes, and contains some results on numbers in geometric progression. Book IX contains a number of theorems of special interest, the pearl being Euclid's celebrated proof of the existence of infinitely many primes, which we shall present in the next unit.

5.2 The Euclidean Algorithm

We have seen that the problem of finding least common multiples can readily be solved by first finding the corresponding greatest common divisor. However, as yet we have no systematic method of finding greatest common divisors. Given two positive numbers a and b we could determine a list of all divisors for each and select the largest one in common. That's fine for reasonably small integers, but would you want to tackle finding $\gcd(3108, 5291)$ by such an approach? There is a more efficient method which uses nothing more than repeated division as in the Division Algorithm; it is called the *Euclidean Algorithm*. First we shall study a numerical example illustrating the procedure before returning to see why it works.

Example 5.1

Determine $\gcd(3108, 5291)$.

We begin by dividing the larger number by the smaller.

$$5291 = 1 \times 3108 + 2183$$

Next we divide 3108 by the remainder in the above equation.

$$3108 = 1 \times 2183 + 925$$

We continue in this fashion, at each stage dividing the previous divisor by the resulting remainder.

$$\begin{aligned} 2183 &= 2 \times 925 + 333 \\ 925 &= 2 \times 333 + 259 \\ 333 &= 1 \times 259 + 74 \\ 259 &= 3 \times 74 + 37 \\ 74 &= 2 \times 37 + 0 \end{aligned}$$

The process stops when a remainder of 0 is encountered and the last non-zero remainder, namely 37, is the required $\gcd(3108, 5291)$. ♦

Let us spell out generally the system of equations that arise in applying the Euclidean Algorithm to determine $\gcd(a, b)$. Notice that since $\gcd(-a, b) = \gcd(a, b)$ we can safely assume that the numbers in question are both positive. Repeated application of the Division Algorithm gives the following.

$$\begin{array}{lll} a & = q_1 \times b + r_1 & 0 < r_1 < b \\ b & = q_2 \times r_1 + r_2 & 0 < r_2 < r_1 \\ r_1 & = q_3 \times r_2 + r_3 & 0 < r_3 < r_2 \\ r_2 & = q_4 \times r_3 + r_4 & 0 < r_4 < r_3 \\ & \cdots & \\ r_{n-2} & = q_n \times r_{n-1} + r_n & 0 < r_n < r_{n-1} \\ r_{n-1} & = q_{n+1} \times r_n + 0 & \end{array}$$

The first fact to observe is that b and the remainders form a strictly decreasing sequence of positive integers.

$$b > r_1 > r_2 > \cdots > r_{n-2} > r_{n-1} > r_n > 0$$

This establishes that the algorithm must terminate; a zero remainder must be reached in a finite number of steps.

The outstanding claim to be justified is that the final non-zero remainder, namely r_n, is equal to the greatest common divisor of the two original numbers, a and b. This is an immediate consequence of the following simple property of greatest common divisors.

> If $a = qb + r$, then $\gcd(a, b) = \gcd(b, r)$.

Why is this? As $\gcd(a, b)$ divides both a and b, it also divides $a - qb$. That is, $\gcd(a, b)$ divides r and, being a common divisor of b and r,

$$\gcd(a, b) \leq \gcd(b, r).$$

But the argument works conversely as well: $\gcd(b, r)$ divides both b and $bq + r$ and so is a common divisor of a and b. Hence

$$\gcd(b, r) \leq \gcd(a, b).$$

From the two inequalities, it follows that

$$\gcd(b, r) = \gcd(a, b).$$

If we apply this fact to the equations in the Euclidean Algorithm system we get:

$$\gcd(a, b) = \gcd(b, r_1) = \gcd(r_1, r_2) = \cdots = \gcd(r_{n-1}, r_n) = \gcd(r_n, 0).$$

Therefore,

$$\gcd(a, b) = \gcd(r_n, 0) = r_n.$$

The system of equations that arise in the Euclidean Algorithm do more for us than just lead to the greatest common divisor of numbers a and b. Theorem 4.3 tells us that $\gcd(a, b)$ can be expressed as an integer combination of a and b, and we are at last in a position to see how to find this expression.

Example 5.2

Find integers m and n such that

$$\gcd(3108, 5291) = m \times 3108 + n \times 5291.$$

Consider the system of equations in Example 5.1 which led to $\gcd(3108, 5291) = 37$. We take these equations in reverse order, starting at the penultimate one which gives the following expression for 37.

$$37 = 259 - 3 \times 74$$

The preceding equation gives an expression for the previous remainder, $74 = 333 - 259$, which when substituted into our expression for 37 gives the following.

$$37 = 259 - 3 \times (333 - 259) = -3 \times 333 + 4 \times 259$$

We continue to step back through the system of equations, at each stage substituting for the remainder in the developing expression for 37.

$$\begin{aligned} 37 &= -3 \times 333 + 4 \times (925 - 2 \times 333) = 4 \times 925 - 11 \times 333 \\ &= 4 \times 925 - 11 \times (2183 - 2 \times 925) = -11 \times 2183 + 26 \times 925 \\ &= -11 \times 2183 + 26 \times (3108 - 2183) = 26 \times 3108 - 37 \times 2183 \\ &= 26 \times 3108 - 37 \times (5291 - 3108) = -37 \times 5291 + 63 \times 3108 \end{aligned}$$

For ease of working with the original equations we shall always write the integer combination at each stage with the larger number first.

Hence we have

$$37 = 63 \times 3108 + (-37) \times 5291,$$

and so $m = 63$ and $n = -37$ are suitable values.

It must be remarked that this is not the only solution. For example, by adding and subtracting 3108×5291 we get

$$\begin{aligned} 37 &= 63 \times 3108 + (-37) \times 5291 + 3108 \times 5291 - 3108 \times 5291 \\ &= 5354 \times 3108 + (-3145) \times 5291, \end{aligned}$$

giving $m = 5354$ and $n = -3145$ as another of infinitely many possibilities. ♦

Problem 5.1

Use the Euclidean Algorithm to determine $\gcd(1092, 777)$ and hence find integers m and n such that

$$\gcd(1092, 777) = 1092m + 777n.$$

5.3 Linear Diophantine equations

Consider the following problem:

> I spent exactly £30 buying some 19p stamps and some 25p stamps so that the difference in number between the two kinds of stamp is as small as possible. How many stamps did I buy?

Translating the information in this problem into an equation to be solved presents little difficulty. Suppose that I bought x stamps at 19p and y stamps at 25p. Then, in all, I shall have spent $19x + 25y$ pence and so

$$19x + 25y = 3000.$$

This is a linear equation in two variables and the problem is asking us to find one particular solution of it. Now in principle, for any value of x we can solve the equation for a corresponding value of y. However there is a major constraint in our problem: we are only interested in solutions of this equation in which the x and y values are non-negative integers, that is, for which $x \geq 0$ and $y \geq 0$. We are not going to be buying either parts of stamps or negative numbers of stamps! It is certainly not the case that for each non-negative integer x there is a corresponding non-negative integer y solving the equation, so there is more to this problem than elementary algebra.

The term *Diophantine equation* is reserved for algebraic equations for which we seek integer solutions. The Diophantine equation arising from the posed problem is one of the form $ax + by = c$ where a, b and c are integers, a and b not both zero. This latter equation is the general *linear Diophantine equation.* In Example 5.2 and Problem 5.1 we have witnessed some instances of linear Diophantine equations and made use of the Euclidean Algorithm in solving them. We shall now build on the techniques of those examples to discover how to solve the linear Diophantine equation in general.

We shall return to the posed question at the end of this section, but let us start with a linear Diophantine equation involving smaller coefficients. Consider the equation $6x + 15y = 4$. Allowing x and y to take all possible values we see that the left-hand side of this equation takes all integer combinations of 6 and 15, which we know to be the set of all multiples of $\gcd(6, 15)$. So the left-hand side takes values in the set

$$\{\ldots, -12, -9, -6, -3, 0, 3, 6, 9, 12, \ldots\}.$$

As 4 does not belong to this set, the Diophantine equation $6x + 15y = 4$ has no solutions. On the other hand the Diophantine equation $6x + 15y = 9$ is seen to have solutions, because 9 is in the set. But notice that this latter equation can be simplified by cancelling the common divisor 3 through the equation to obtain $2x + 5y = 3$. Looked at in this simpler form, $\gcd(2, 5) = 1$ and so the values taken by the left-hand side are all the multiples of 1, which certainly includes 3.

The posed problem involves slightly more than a linear Diophantine equation since it requests solutions which are non-negative.

Consider now the general linear Diophantine equation $ax + by = c$. As x and y range through all integers, the values taken by the left-hand side are all the multiples of $\gcd(a, b)$. So a necessary and sufficient condition for the equation to have solutions is that $\gcd(a, b)$ is a divisor of c.

Suppose that this condition holds. Then each of a, b and c is divisible by $\gcd(a, b)$ and so, if $\gcd(a, b) > 1$, we divide each coefficient by $d = \gcd(a, b)$ to get the 'same' linear Diophantine equation

$$\frac{a}{d}x + \frac{b}{d}y = \frac{c}{d},$$

where

$$\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1.$$

Hence, under the assumption that the linear Diophantine equation is in its lowest terms, (with any common divisor of a, b and c already having been cancelled out), the equation $ax + by = c$ has solutions if, and only if, $\gcd(a, b) = 1$.

Let us look at an example.

Example 5.3

Solve the linear Diophantine equation

$$66x + 40y = 6.$$

As $\gcd(66, 40) = 2$, and 2 divides 6, we first cancel this common divisor to get the 'same' equation

$$33x + 20y = 3.$$

As $\gcd(33, 20) = 1$ this Diophantine equation is known to have solutions.

We apply the Euclidean Algorithm to the pair of coefficients 33 and 20.

$$\begin{aligned} 33 &= 1 \times 20 + 13 \\ 20 &= 1 \times 13 + 7 \\ 13 &= 1 \times 7 + 6 \\ 7 &= 1 \times 6 + 1 \\ 6 &= 6 \times 1 + 0 \end{aligned}$$

This confirms that $\gcd(33, 20) = 1$.

Reversing the equations we obtain the following.

$$\begin{aligned} 1 &= 7 - 6 \\ &= 7 - (13 - 7) = -1 \times 13 + 2 \times 7 \\ &= -1 \times 13 + 2 \times (20 - 13) = 2 \times 20 - 3 \times 13 \\ &= 2 \times 20 - 3 \times (33 - 20) = -3 \times 33 + 5 \times 20 \end{aligned}$$

This expresses 1 as an integer combination of 33 and 20, but our present task is to express 3 in this way. Multiplying through by 3 we get

$$3 = -9 \times 33 + 15 \times 20,$$

so that $x = -9$, $y = 15$ is one solution of $33x + 20y = 3$.

Now what other solutions are there? Suppose that $x = -9 + s$, $y = 15 - t$ is another solution, for some integers s and t. Then

$$33(-9 + s) + 20(15 - t) = 3,$$

which simplifies to

$$33s = 20t.$$

We could equally well have taken $y = 15 + t$ which might seem more obvious. However this expression is chosen to conform with the usual general solution given below.

This equation carries the implication that 20 divides $33s$, and, as $\gcd(20, 33) = 1$, Euclid's Lemma gives that 20 divides s. So we can put $s = 20k$, for some integer k, and then from

$$33 \times 20k = 20t$$

we deduce that $t = 33k$.

This argument has shown that any solution must take the form

$$x = -9 + 20k, \quad y = 15 - 33k, \quad \text{for some integer } k,$$

and, moreover, substituting back into the original equation confirms that, for any integer k, this is indeed a solution. So the general solution of this linear Diophantine equation is

$$x = -9 + 20k, \quad y = 15 - 33k, \quad k \in \mathbb{Z}. \qquad \blacklozenge$$

We have illustrated the following.

Theorem 5.1 Solution of linear Diophantine equations

The linear Diophantine equation $ax + by = c$ has solutions if, and only if, $\gcd(a, b)$ divides c.

If this condition holds with $\gcd(a, b) > 1$ then division by $\gcd(a, b)$ simplifies the equation to $a'x + b'y = c'$, where $\gcd(a', b') = 1$.

If x_0, y_0 is one solution of this equation then the general solution is

$$x = x_0 + b'k, \quad y = y_0 - a'k, \quad k \in \mathbb{Z}.$$

The equation is trivial when either a or b is zero, so we assume that both a and b are non-zero.

Proof of Theorem 5.1

The only assertion in the theorem that has not already been verified is that concerning the general solution. To that end suppose that x', y' is any solution of the equation. Then we have

$$a'x_0 + b'y_0 = c' \quad \text{and} \quad a'x' + b'y' = c',$$

and on substituting for c' and rearranging,

$$a'(x' - x_0) = b'(y_0 - y').$$

This equation shows that a' divides $b'(y_0 - y')$ and since $\gcd(a', b') = 1$, Euclid's Lemma gives that a' divides $(y_0 - y')$. That is, $y_0 - y' = a'k$ for some integer k. Substitution of this into the above equation then gives $x' - x_0 = b'k$, and so

$$x' = x_0 + b'k, \quad y' = y_0 - a'k, \quad \text{for some integer } k,$$

as claimed.

Finally, a simple substitution shows that integers

$$x_0 + b'k \quad \text{and} \quad y_0 - a'k$$

constitute a solution of $a'x + b'y = c'$ for any integer k. ■

The two problems which follow require you to find the general solution of a linear Diophantine equation but each then asks you to find, within the general solution, one particular solution.

Problem 5.2

Find the general solution of the linear Diophantine equation

$$45x - 14y = 7$$

and also the particular solution in which x takes its least positive value.

Problem 5.3

I spent exactly £30 buying some 19p stamps and some 25p stamps so that the difference in number between the two kinds of stamps is as small as possible. How many stamps did I buy?

As the course progresses we shall discover that linear Diophantine equations crop up frequently, in many guises, often as subsidiary tasks in some deeper problem. When we next meet a linear Diophantine equation we shall develop some improved techniques for solving them, removing the need to fall back on the Euclidean Algorithm.

ADDITIONAL EXERCISES

Section 2

1 In Problem 2.5 we discovered the formula

$$P(k,n) = P(k-1,n) + P(3,n-1).$$

Examination of the table of pyramidal numbers shows that the same relationship appears to hold there, namely that each entry in the body of the table is equal to the one above it added to the triangle-based pyramidal number at the top of the previous column. That is,

$$Q(k,n) = Q(k-1,n) + Q(3,n-1).$$

Assuming this formula is true, show that

$$Q(5,n) = \tfrac{1}{2}n^2(n+1)$$

and determine $Q(6,n)$.

2 If either $2n^2+1$ or $2n^2-1$ is a square, say m^2, show that $(nm)^2$ is a triangular number. Hence find three examples of squares which are also triangular.

3 Show that no triangular number can have 2, 4, 7 or 9 as its last digit.

Section 3

1 Use Mathematical Induction to prove the following formulae are true for all positive integers n.

(a) $1^2+3^2+5^2+\cdots+(2n-1)^2 = \frac{1}{3}n(4n^2-1)$

(b) $1+5+12+22+\cdots+\frac{1}{2}n(3n-1) = \frac{1}{2}n^2(n+1)$

Note that part (b) establishes the formula for sums of pentagonal numbers derived as $Q(5,n)$ in Exercise 1 of Section 2 above.

2 Prove the following formula for all positive integers n.

$$1\times(1!)+2\times(2!)+3\times(3!)+\cdots+n\times(n!) = (n+1)!-1$$

3 For which positive integers n is it true that $n! > 6n^2$? Prove your result using Mathematical Induction.

4 Show how the Second Principle of Mathematical Induction can be derived from the Well-Ordering Principle, as follows:

Suppose that the conditions (a) and (b′) hold (see page 18), and let S be the set of integers k, $k \geq n_0$, for which $P(k)$ does not hold, that is,

$$S = \{k \in \mathbb{Z} : k \geq n_0 \text{ and } P(k) \text{ is false}\}.$$

Suppose that S is non-empty and use this assumption to deduce that the set

$$T = \{s - n_0 : s \in S\}$$

is a non-empty set of positive integers. Show how the existence of a least member of T leads to a contradiction.

Section 4

1 Show that an integer of the form $12k + 8$ is necessarily of the form $3m + 2$, but not conversely.

For which values of r in the range $0 \leq r \leq 11$ is $12k + r$ also of the form $3m + 2$?

2 The number 64 is both a square and a cube:

$$64 = 8^2 = 4^3.$$

Show that any number which is both a square and a cube must be of one of the forms $7k$ or $7k + 1$.

3 Use Mathematical Induction to prove that 9 divides $4^n + 6n - 1$ for all integers $n \geq 0$.

4 Note that

$$36 \mid 108, \quad 36 \mid 1008, \quad 36 \mid 10\,008, \quad 36 \mid 100\,008, \quad \ldots.$$

Formulate the general result illustrated by this and prove your result using Mathematical Induction.

5 Prove that if $\gcd(a, b) = 1$ then, for any integer c,

$$\gcd(ac, b) = \gcd(c, b).$$

6 Suppose that $\gcd(a, b) = 1$. Show that:

(a) $\gcd(2a + b, a + b) = 1$;

(b) $\gcd(a + b, a - b) = 1$ or 2.

Section 5

1 Use the Euclidean Algorithm to find integers x and y satisfying

$$\gcd(17, 143) = 17x + 143y.$$

Give the general solution of this equation and the particular solution in which x takes its least positive value.

2 Find all solutions in positive integers of the Diophantine equation

$$158x - 57y = 7.$$

3 Using only 2p, 10p and 50p coins, in how many ways can exactly 100 coins be used to total £6.

Challenge Problems

1 The triangular numbers 15 and 21 have the property that their sum, 36, and their difference, 6, are also triangular. Find another pair of triangular numbers with this property.

[There is one other pair both of which are less than 200, and a third pair both of which are less than 1000.]

2 Find all ways in which the integer 1000 can be written as a sum of consecutive positive integers. For example,

$$1000 = 198 + 199 + 200 + 201 + 202,$$

is one way. Repeat for 1200. Generalize.

3 Which of the following is the larger

$$\frac{(2n)!}{(n!)^2} \quad \text{or} \quad \frac{2^{2n}}{5}?$$

Prove your result by Mathematical Induction.

4 After spilling my coffee over the exact long division shown below, only the middle digit of the quotient is now legible. As it happens, enough information remains for you to reconstruct the long division uniquely.

```
                  x x 8 x x
x x x ) x x x x x x x x
          x x x
            x x x x
              x x x
                x x x x
                x x x x
```

SOLUTIONS TO THE PROBLEMS

Solution 2.1

(a) The first term $a = 3$ and the common difference $d = 7$. As the last term is $143 = 3 + (20 \times 7)$ we see that the number of terms $n = 21$. So the formula for an arithmetic series gives the sum as

$$\tfrac{1}{2} \times 21(6 + (20 \times 7)) = 1533.$$

(b) There are 26 terms averaging $\frac{1}{2}(100 + 50) = 75$ and so the sum is $26 \times 75 = 1950$.

Solution 2.2

If n is the triangular number T_r, then $n = \frac{1}{2}r(r+1)$. Therefore

$$\begin{aligned} 9n + 1 &= \tfrac{1}{2} \times 9r(r+1) + 1 \\ &= \tfrac{1}{2}(9r^2 + 9r + 2) \\ &= \tfrac{1}{2}(3r+1)(3r+2) \\ &= \tfrac{1}{2}k(k+1), \quad \text{where } k = 3r + 1, \end{aligned}$$

which is the triangular number T_k.

Solution 2.3

First we show that if n is triangular number then $8n + 1$ is square. So let n be the triangular number T_r, namely $n = \frac{1}{2}r(r+1)$. Therefore

$$8n + 1 = \tfrac{1}{2} \times 8r(r+1) + 1 = 4r(r+1) + 1 = (2r+1)^2,$$

which is a square.

Conversely, suppose that $8n + 1$ is a square so that $8n + 1 = k^2$ for some integer k. But $8n + 1$ is odd and so k must be odd (because the square of any even integer is even). So $k = 2s + 1$, for some integer s, and

$$8n + 1 = (2s+1)^2 = 4s(s+1) + 1,$$

which gives $n = \frac{1}{2}s(s+1)$, the triangular number T_s.

The even integers

$$\ldots, -4, -2, 0, 2, 4, \ldots$$

are those which can be written as $2m$, for some integer m.

The odd integers

$$\ldots, -3, -1, 1, 3, 5, \ldots$$

are those which can be written as $2m + 1$, for some integer m.

Solution 2.4

The nth k-gonal number is the sum of n terms of the arithmetic progression with first term $a = 1$ and common difference $d = k - 2$. So, by the formula for arithmetic series,

$$P(k, n) = \tfrac{1}{2}n(2 + (n-1)(k-2)).$$

To get the right-hand column of the table of polygonal numbers we put $k = 3, 4, 5, \ldots, 8$ giving respectively

$$\tfrac{1}{2}n(n+1),\ n^2,\ \tfrac{1}{2}n(3n-1),\ n(2n-1),\ \tfrac{1}{2}n(5n-3),\ n(3n-2).$$

To get the k-gonal numbers in the bottom row of the table we put $n = 1, 2, 3, \ldots, 8$ giving respectively

$$1,\ k,\ 3(k-1),\ 2(3k-4),\ 5(2k-3),\ 3(5k-8),\ 7(3k-5),\ 4(7k-12).$$

Solution 2.5

Using the formula from Problem 2.4 we obtain the following.

$$\begin{aligned} P(k-1, n) + P(3, n-1) &= \tfrac{1}{2}n(2 + (n-1)(k-3)) + \tfrac{1}{2}(n-1)n \\ &= \tfrac{1}{2}n(2 + nk - k - 3n + 3 + n - 1) \\ &= \tfrac{1}{2}n(4 + nk - k - 2n) \\ &= \tfrac{1}{2}n((n-1)(k-2) + 2) \\ &= P(k, n) \end{aligned}$$

Solution 3.1

(a) (i) The set $\{n \in \mathbb{Z} : |n| > 20\}$ consists of all the integers except those from -20 to 20 inclusive. That is

$$\{\ldots, -23, -22, -21, 21, 22, 23, \ldots\}.$$

(ii) The set $\{n : n = 2m^2, \text{ for some } m \in \mathbb{Z}^+\}$, is the set consisting of those integers which are equal to twice the square of a positive integer, namely

$$\{2, 8, 18, 32, \ldots\}.$$

(b) (i) The set is $\{n : n = 2m + 1, \text{ for some } m \in \mathbb{Z},\ m \geq 0\}$.

(ii) The set is $\{n \in \mathbb{Z} : |n| \leq 100\}$.

We could equally well write the set in part (i) as $\{n : n = 2m - 1,\ m \in \mathbb{Z}^+\}$.

Solution 3.2

Let $P(n)$ be the proposition that the formula is true for the positive integer n.

Then $P(1)$ is true since

$$1^2 = \tfrac{1}{6}(1+1)(2+1).$$

Hence we have the basis for the induction.

For the induction step, assume that $P(k)$ is true, that is,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 = \tfrac{1}{6}k(k+1)(2k+1).$$

Then,

$$\begin{aligned} &1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2, \quad \text{the LHS of } P(k+1), \\ &= \tfrac{1}{6}k(k+1)(2k+1) + (k+1)^2, \quad \text{by the induction hypothesis,} \\ &= \tfrac{1}{6}(k+1)\left(2k^2 + k + 6(k+1)\right) \\ &= \tfrac{1}{6}(k+1)\left(2k^2 + 7k + 6\right) \\ &= \tfrac{1}{6}(k+1)(k+2)(2k+3) \\ &= \tfrac{1}{6}(k+1)((k+1)+1)(2(k+1)+1), \quad \text{the RHS of } P(k+1). \end{aligned}$$

This shows that $P(k+1)$ is true and completes the induction step.

Hence, by Mathematical Induction, the formula is true for all positive integers n.

Solution 3.3

Let $P(n)$ be the proposition that the formula is true for n.

Then $P(1)$ is true since

$$a = \frac{a(r-1)}{r-1}.$$

So we have the basis for the induction.

Now for the induction step. Assume that $P(k)$ is true; that is,

$$a + ar + ar^2 + ar^3 + \cdots + ar^{k-1} = \frac{a(r^k - 1)}{r-1}.$$

Then,

$$\begin{aligned} a + ar + ar^2 + ar^3 + \cdots + ar^{k-1} + ar^k \\ &= \frac{a(r^k-1)}{r-1} + ar^k, \quad \text{by the induction hypothesis,} \\ &= \frac{a(r^k-1) + ar^k(r-1)}{r-1} \\ &= \frac{a(r^{k+1}-1)}{r-1}, \end{aligned}$$

which is $P(k+1)$, completing the induction step.

Hence, by Mathematical Induction the formula is true for all positive integers n.

Solution 3.4

Let $P(n)$ be the proposition that $L_n < \left(\frac{7}{4}\right)^n$.

We note that $P(1)$ and $P(2)$ are true since

$$L_1 = 1 < \tfrac{7}{4} \quad \text{and} \quad L_2 = 3 < \left(\tfrac{7}{4}\right)^2 = \tfrac{49}{16}.$$

So we have the basis for the induction.

For the induction step (using the Second Principle) suppose that $P(1)$, $P(2)$, ..., $P(k)$ are all true and consider $P(k+1)$ for $k \geq 2$.

$$\begin{aligned} L_{k+1} &= L_k + L_{k-1} \\ &< \left(\tfrac{7}{4}\right)^k + \left(\tfrac{7}{4}\right)^{k-1}, \quad \text{by the induction hypothesis,} \\ &= \left(\tfrac{7}{4}\right)^{k-1}\left(\tfrac{7}{4}+1\right) = \left(\tfrac{7}{4}\right)^{k-1}\left(\tfrac{11}{4}\right) \\ &< \left(\tfrac{7}{4}\right)^{k-1}\left(\tfrac{7}{4}\right)^2 = \left(\tfrac{7}{4}\right)^{k+1}, \quad \text{since } \tfrac{11}{4} < \tfrac{49}{16}, \end{aligned}$$

which shows that $P(k+1)$ is true.

Hence, by the Second Principle of Mathematical Induction, the proposition is true.

Note that, since the proof of the induction step uses the formula $L_{k+1} = L_k + L_{k-1}$, which holds only for $k \geq 2$, our induction step first deduces the truth of $P(3)$ from that of $P(1)$ and $P(2)$, and then goes on to deduce the truth of $P(4)$, $P(5)$, etc. It does not deduce the truth of $P(2)$ from $P(1)$. Hence we have to show the truth of $P(2)$ separately, which we have done by including it in the basis for the induction.

Solution 4.1

Cubing each of the given forms that the number can take on dividing by 3, and writing each in the form $9n + r$, where $0 \leq r \leq 8$ we get:

$$(3n)^3 = 27n^3 = 9(3n^3) + 0;$$
$$(3n+1)^3 = 27n^3 + 27n^2 + 9n + 1 = 9(3n^3 + 3n^2 + n) + 1;$$
$$(3n+2)^3 = 27n^3 + 54n^2 + 36n + 8 = 9(3n^3 + 6n^2 + 4n) + 8.$$

Hence the only possible remainders are 0, 1 and 8.

Solution 4.2

The divisors of 18 are 1, 2, 3, 6, 9 and 18. The divisors of -24 are 1, 2, 3, 4, 6, 8, 12 and 24. (These are the same as the divisors of 24.)

The common divisors of 18 and -24 are 1, 2, 3 and 6.

Solution 4.3

The Division Algorithm tells us that n takes one of the three forms: $3k$, $3k+1$ or $3k+2$. But n is not a multiple of 3 and so the first of these forms cannot occur.

For $n = 3k+1$ we get

$$(3k+1)^2 - 1 = 9k^2 + 6k + 1 - 1 = 3(3k^2 + 2k).$$

For $n = 3k+2$ we get

$$(3k+2)^2 - 1 = 9k^2 + 12k + 4 - 1 = 3(3k^2 + 4k + 1).$$

In each case $n^2 - 1$ has remainder zero on dividing by 3.

Solution 4.4

(a) First let m and n be integers such that $\gcd(a,b) = ma + nb$. Then, if c is a multiple of $\gcd(a,b)$, say $c = k \times \gcd(a,b)$, we have

$$c = k(ma + nb) = (km)a + (kn)b,$$

showing that c is an integer combination of a and b.

Remember that 'if, and only if,' proofs have two parts.

Conversely, if there are integers x and y such that $c = xa + yb$ then, as $\gcd(a,b)$ divides both a and b, property (e) of Theorem 4.2 confirms that $\gcd(a,b)$ divides $c = xa + yb$.

(b) Let $d = \gcd(a,b)$. Then, from $d = ma + nb$, we have $1 = ma' + nb'$, where $a = da'$ and $b = db'$. It follows from Theorem 4.4 that $\gcd(m,n) = 1$.

(c) From $\gcd(a,b) = 1$ and $\gcd(a,c) = 1$ we infer the existence of integers m, n, r and s such that

$$ma + nb = 1 \quad \text{and} \quad ra + sc = 1.$$

Multiplying these two equations together we get

$$(ma + nb)(ra + sc) = 1,$$

and collating the terms appropriately we get

$$(mar + nbr + msc)a + (ns)bc = 1,$$

which expresses 1 as an integer combination of a and bc. The result then follows from Theorem 4.4.

(d) There exist integers m, n, r and s such that

$$\gcd(a,b) = ma + nb \quad \text{and} \quad \gcd(ka,kb) = r(ka) + s(kb).$$

But then

$$\begin{aligned}\gcd(ka, kb) &= k(ra + sb) \\ &= k(t \times \gcd(a, b)), \quad \text{for some integer } t, \text{ by part (a) above,} \\ &= t(k \times \gcd(a, b)).\end{aligned}$$

In addition we have

$$\begin{aligned}k \times \gcd(a, b) &= k(ma + nb) \\ &= m(ka) + n(kb) \\ &= u \times \gcd(ka, kb), \quad \text{for some integer } u, \text{ by part (a).}\end{aligned}$$

We now know that each of $\gcd(ka, kb)$ and $k \times \gcd(a, b)$ is a multiple of the other, and so the equality follows from property (c) of Theorem 4.2.

Solution 4.5

As $\gcd(a, b) = 1$ there exist integers m and n such that

$$ma + nb = 1.$$

(a) If c divides a then $a = kc$ for some integer k. Substituting for this value of a we get

$$mkc + nb = 1$$

and, as this expresses 1 as an integer combination of b and c, Theorem 4.4 gives $\gcd(b, c) = 1$.

(b) Our goal is to express 1 as an integer combination of a and $a + b$. This is achieved from the equation $ma + nb = 1$ by careful rearrangement:

$$ma + nb = (m - n)a + n(a + b) = 1.$$

Solution 4.6

(a) If d divides $3n + 2$ and d divides $5n + 3$ then d divides $5(3n + 2) + (-3)(5n + 3)$. That is, d divides 1. It follows that the only common divisor of $3n + 2$ and $5n + 3$ is 1.

(b) Let $\gcd(n^2 + 3n + 1, 2n + 1) = d$. Then d divides any integer combination of $n^2 + 3n + 1$ and $2n + 1$. We first choose one which eliminates the term in n^2, namely

$$2(n^2 + 3n + 1) + (-n)(2n + 1) = 5n + 2.$$

So d divides $5n + 2$ and d divides $2n + 1$. Therefore, as before, d divides any integer combination of $5n + 2$ and $2n + 1$. This time we choose one which eliminates the term in n, namely

$$5(2n + 1) + (-2)(5n + 2) = 1.$$

We conclude that $d = 1$.

Solution 4.7

Let $\text{lcm}(a, b) = l$ and let m be any common multiple of a and b. Using the Division Algorithm for dividing m by l, there exist integers q and r, such that

$$m = lq + r, \quad \text{where } 0 \leq r < l.$$

Rearranging gives

$$r = 1m + (-q)l$$

and, as both m and l are common multiples of a and b, so too is the integer combination r. But l is the least common multiple of a and b, so we cannot have $0 < r < l$.

Hence $r = 0$, $m = ql$ and so l divides m.

Solution 4.8

(a) Since $\gcd(18, 24) = 6$, $\operatorname{lcm}(18, 24) = \dfrac{18 \times 24}{6} = 72$.

(b) The divisors of 27 are 1, 3, 9 and 27 and the largest of these which is also a divisor of 45 is 9. Therefore $\gcd(27, 45) = 9$ and $\operatorname{lcm}(27, 45) = \dfrac{27 \times 45}{9} = 135$.

(c) Any common divisor of 39 and 40 must divide $40 - 39$. It follows that $\gcd(39, 40) = 1$ and $\operatorname{lcm}(39, 40) = 39 \times 40 = 1560$.

Note that this example illustrates the general result that, for positive integers a and b,

$$\operatorname{lcm}(a, b) = ab \text{ if, and only if, } \gcd(a, b) = 1.$$

Solution 5.1

$$\begin{aligned} 1092 &= 1 \times 777 + 315 \\ 777 &= 2 \times 315 + 147 \\ 315 &= 2 \times 147 + 21 \\ 147 &= 7 \times 21 + 0 \end{aligned}$$

So $\gcd(1092, 777)$ is equal to the last non-zero remainder, namely 21.

Reversing the equations we get the following.

$$\begin{aligned} \gcd(1092, 777) = 21 &= 315 - 2 \times 147 \\ &= 315 - 2 \times (777 - 2 \times 315) = -2 \times 777 + 5 \times 315 \\ &= -2 \times 777 + 5 \times (1092 - 777) = 5 \times 1092 - 7 \times 777 \end{aligned}$$

The required integers are $m = 5$ and $n = -7$.

Solution 5.2

The negative coefficient causes no problem if we think of the equation as being $45x + 14(-y) = 7$ and solve for x and $-y$. The Euclidean Algorithm yields the following equations.

$$\begin{aligned} 45 &= 3 \times 14 + 3 \\ 14 &= 4 \times 3 + 2 \\ 3 &= 1 \times 2 + 1 \\ 2 &= 2 \times 1 + 0 \end{aligned}$$

Reversing the equations we have

$$\begin{aligned} 1 &= 3 - 2 = 3 - (14 - 4 \times 3) = -1 \times 14 + 5 \times 3 \\ &= -1 \times 14 + 5 \times (45 - 3 \times 14) = 5 \times 45 - 16 \times 14. \end{aligned}$$

Multiplying by 7,

$$7 = 35 \times 45 - 112 \times 14,$$

so that $x = 35$, $-y = -112$, (that is $y = 112$) is one solution of the equation.

The general solution is

$$x = 35 + 14k, \quad y = 112 + 45k, \quad k \in \mathbb{Z}.$$

The expression for y comes from our previous general solution which gave $-y = -112 - 45k$. If you substitute back into the original equation, note that these 'added' terms do cancel.

The solution with the least positive value of x is given by $k = -2$, namely $x = 7$, $y = 22$.

Note that the general solution can therefore be written as

$$x = 7 + 14k, \quad y = 22 + 45k, \quad k \in \mathbb{Z}.$$

Solution 5.3

If x is the number of 19p stamps bought and y is the number of 25p stamps then the total amount spent is $19x + 25y$ pence. Hence $19x + 25y = 3000$, and, as we are not allowed fractions of stamps, our task is to solve this in integers.

We first note that $\gcd(19, 25) = 1$, so there are solutions. It is not difficult to 'spot' one solution of this equation. As 25 divides 3000 we see that $x = 0$, $y = 120$ is a solution. Theorem 5.1 then gives the general solution:

$$x = 0 + 25k, \quad y = 120 - 19k, \quad k \in \mathbb{Z}.$$

You can, of course, use the Euclidean Algorithm, but do look out for short cuts.

We seek the particular solution in which $|y - x|$ is as small as possible; that is we want to minimize $|120 - 44k|$. But $44k$ is nearest to 120 when $k = 3$, giving $x = 75$, $y = 63$ as the required solution.

SOLUTIONS TO ADDITIONAL EXERCISES

Section 2

1 Putting $k = 5$ we have

$$Q(5, n) = Q(4, n) + Q(3, n - 1).$$

That is, from the table of pyramidal numbers,

$$\begin{aligned} Q(5, n) &= \tfrac{1}{6}n(n + 1)(2n + 1) + \tfrac{1}{6}(n - 1)(n)(n + 1) \\ &= \tfrac{1}{6}n(n + 1)(2n + 1 + n - 1) \\ &= \tfrac{1}{6}n(n + 1)(3n) \\ &= \tfrac{1}{2}n^2(n + 1). \end{aligned}$$

Putting $k = 6$ we have

$$Q(6, n) = Q(5, n) + Q(3, n - 1),$$

and so

$$\begin{aligned} Q(6, n) &= \tfrac{1}{2}n^2(n + 1) + \tfrac{1}{6}(n - 1)(n)(n + 1) \\ &= \tfrac{1}{6}n(n + 1)(3n + n - 1) \\ &= \tfrac{1}{6}n(n + 1)(4n - 1). \end{aligned}$$

2 If $2n^2 \pm 1 = m^2$ then

$$\begin{aligned} (nm)^2 = n^2m^2 &= n^2(2n^2 \pm 1) \\ &= \tfrac{1}{2}2n^2(2n^2 \pm 1) \\ &= \tfrac{1}{2}k(k + 1), \quad \text{where } k = 2n^2 \text{ when } 2n^2 + 1 \text{ is square,} \\ &\qquad\qquad \text{or } k = 2n^2 - 1 \text{ when } 2n^2 - 1 \text{ is square,} \\ &= T_k. \end{aligned}$$

The first three positive values of n for which either $2n^2 + 1$ or $2n^2 - 1$ is square are:

$n = 1$, $2n^2 - 1 = 1$, giving $T_1 = \frac{1}{2} \times 1 \times 2 = 1$;

$n = 2$, $2n^2 + 1 = 9$, giving $T_8 = \frac{1}{2} \times 8 \times 9 = 36$;

$n = 5$, $2n^2 - 1 = 49$, giving $T_{49} = \frac{1}{2} \times 49 \times 50 = 1225$.

If you overlooked $n = 1$, the next occurs at $n = 12$ giving $T_{288} = 41\,616$ as a triangular square.

3 Any triangular number is of the form $\frac{1}{2}k(k+1)$. If the final digit of such a number was any of 2, 4, 7 or 9 then, by doubling, the final digit of $k(k+1)$ would be 4, 8, 4 or 8 respectively. Now k can take any positive value so its final digit can be any of the ten digits. In each case we can determine the final digit of $k(k+1)$, as follows.

Final digit of k	Final digit of $k+1$	Final digit of $k(k+1)$
0	1	0
1	2	2
2	3	6
3	4	2
4	5	0
5	6	0
6	7	2
7	8	6
8	9	2
9	0	0

This confirms that the final digit of $k(k+1)$ cannot be 4 or 8 and so the final digit of $\frac{1}{2}k(k+1)$ cannot be 2, 4, 7 or 9.

Section 3

1 (a) Let $P(n)$ be the proposition that the formula is true for n.

$P(1)$ is true, since

$$1^2 = \tfrac{1}{3} \times 1 \times (4 \times 1^2 - 1).$$

This gives the basis for the induction.

Assume that $P(k)$ is true for some positive integer k; that is,

$$1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \tfrac{1}{3}k(4k^2 - 1).$$

Then,

$$\begin{aligned}
&1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 + (2k+1)^2 \\
&\quad = \tfrac{1}{3}k(4k^2 - 1) + (2k+1)^2 \\
&\quad = \tfrac{1}{3}k(2k-1)(2k+1) + (2k+1)^2 \\
&\quad = \tfrac{1}{3}(2k+1)(k(2k-1) + 3(2k+1)) \\
&\quad = \tfrac{1}{3}(2k+1)(2k^2 + 5k + 3) \\
&\quad = \tfrac{1}{3}(2k+1)(2k+3)(k+1) \\
&\quad = \tfrac{1}{3}(k+1)(4k^2 + 8k + 3) \\
&\quad = \tfrac{1}{3}(k+1)(4(k+1)^2 - 1),
\end{aligned}$$

which establishes the truth of $P(k+1)$, and completes the induction step.

Hence, by the Principle of Mathematical Induction, the formula is true for all positive integers.

(b) Let $P(n)$ be the proposition that the formula is true for n.

Then $P(1)$ is true since

$$1 = \tfrac{1}{2} \times 1^2(1+1).$$

Assume that $P(k)$ is true; that is

$$1 + 5 + 12 + 22 + \cdots + \tfrac{1}{2}k(3k-1) = \tfrac{1}{2}k^2(k+1).$$

Then,

$$\begin{aligned}1+5+12+22+\cdots+\tfrac{1}{2}k(3k-1)+\tfrac{1}{2}(k+1)(3k+2)&\\ =\tfrac{1}{2}k^2(k+1)+\tfrac{1}{2}(k+1)(3k+2)&\\ =\tfrac{1}{2}(k+1)(k^2+3k+2)&\\ =\tfrac{1}{2}(k+1)^2(k+2),&\end{aligned}$$

confirming that $P(k+1)$ is true and completing the induction step.

The truth of $P(n)$ for all $n \geq 1$ follows by Mathematical Induction.

2 Let $P(n)$ be the proposition that the formula is true for n.

$P(1)$ is certainly true since

$$1 \times (1!) = 2! - 1,$$

both sides being equal to 1.

For the induction step, suppose that $P(k)$ is true for some $k \geq 1$. That is,

$$1 \times (1!) + 2 \times (2!) + \cdots + k \times (k!) = (k+1)! - 1.$$

Then,

$$\begin{aligned}1 \times (1!) + 2 \times (2!) + \cdots + k \times (k!) + (k+1)((k+1)!)&\\ = (k+1)! - 1 + (k+1)((k+1)!)&\\ = (k+1)!(1+k+1) - 1&\\ = (k+2)! - 1,&\end{aligned}$$

which confirms the truth of $P(k+1)$ and completes the induction step.

The truth of $P(n)$ for all $n \geq 1$ follows by Mathematical Induction.

3 We observe that $n!$ increases more rapidly than $6n^2$ and $n!$ exceeds $6n^2$ for the first time when $n = 6$. So we make the conjecture that $n! > 6n^2$ for all $n \geq 6$, and prove it by Mathematical Induction.

Let $P(n)$ be the proposition that $n! > 6n^2$. Now

$$6! = 720 > 6 \times 6^2 = 216,$$

so $P(6)$ is true, giving us the basis for induction.

Suppose that $P(k)$ is true for some integer $k \geq 6$, that is,

$$k! > 6k^2.$$

On multiplying through by $k+1$, we get

$$(k+1)! > 6k^2(k+1).$$

If we can show that $k^2 \geq k+1$ it will follow that $(k+1)! > 6(k+1)^2$, completing the induction step. But this is easily seen since, for $k \geq 6$,

$$k^2 \geq 6k > 2k = k + k > k + 1,$$

which competes the induction step.

So by Mathematical Induction, $P(n)$ is true for all $n \geq 6$.

4 Suppose that

$$S = \{k \in \mathbb{Z} : k \geq n_0 \text{ and } P(k) \text{ is false}\}$$

is non-empty.

Now consider the set T defined by

$$T = \{s - n_0 : s \in S\}.$$

Certainly T is a non-empty set of integers, each of whose elements is greater than or equal to zero. Now, by (a), $P(n_0)$ is true so n_0 is not an element of S. Hence zero is not an element of T, and T is a non-empty subset of $\mathbb{Z}^+$.

By the Well-Ordering Principle, T has a least element. We may take this least element of T to be $m - n_0$ for some integer m in S and, by the definition of T, it follows that m is the least element of S.

Furthermore, since $n_0 \notin S$ we know that $m > n_0$. By the definition of S it follows that $P(n_0), P(n_0 + 1), \ldots, P(m - 1)$, must all be true. But then (b′) gives that $P(m)$ is true, which contradicts the fact that $m \in S$.

Notice we need the condition that $m > n_0$ to ensure that the list $P(n_0), P(n_0 + 1), \ldots, P(m - 1)$ contains at least one element.

Hence the only assumption made, namely that S is non-empty, must be false. Hence S is empty and so the proposition $P(k)$ is true for all $k \geq n_0$.

Section 4

1 We have,

$$12k + 8 = 3(4k + 2) + 2 = 3m + 2,$$

where $m = 4k + 2$ is an integer.

The converse is false since, for example, $5 = 3 \times 1 + 2$ is of the form $3m + 2$ but has remainder 5 on division by 12.

Since 3 divides 12, the remainder when $12k + r$ is divided by 3 is the same as the remainder when r is divided by 3. This will be equal to 2 when $r = 2$, 5, 8 or 11.

2 Any number is necessarily of one of the forms $7k + r$, where $r = 0$, 1, 2, 3, 4, 5 or 6. Consider the resulting forms for $(7k + r)^2$ and $(7k + r)^3$. Note that

$$(7k + r)^2 = 49k^2 + 14kr + r^2 = 7(7k^2 + 2kr) + r^2$$

and

$$(7k + r)^3 = 343k^3 + 147k^2r + 21kr^2 + r^3 = 7(49k^3 + 21k^2r + 3kr^2) + r^3.$$

Therefore the remainder on dividing $(7k + r)^2$ by 7 is the same as the remainder on dividing r^2 by 7. Similarly, the remainder on dividing $(7k + r)^3$ by 7 is the same as the remainder on dividing r^3 by 7.

$$\begin{array}{ll} 0^2 = 0 = 7 \times 0 + 0 & 0^3 = 0 = 7 \times 0 + 0 \\ 1^2 = 1 = 7 \times 0 + 1 & 1^3 = 1 = 7 \times 0 + 1 \\ 2^2 = 4 = 7 \times 0 + 4 & 2^3 = 8 = 7 \times 1 + 1 \\ 3^2 = 9 = 7 \times 1 + 2 & 3^3 = 27 = 7 \times 3 + 6 \\ 4^2 = 16 = 7 \times 2 + 2 & 4^3 = 64 = 7 \times 9 + 1 \\ 5^2 = 25 = 7 \times 3 + 4 & 5^3 = 125 = 7 \times 17 + 6 \\ 6^2 = 36 = 7 \times 5 + 1 & 6^3 = 216 = 7 \times 30 + 6 \end{array}$$

From the above, we see that the possible remainders on dividing a square by 7 are 0, 1, 2 and 4 and that the possible remainders on dividing a cube by 7 are 0, 1 and 6. So if a number is both a square and a cube then its remainder on division by 7 must be either 0 or 1.

3 Let $P(n)$ be the statement that 9 divides $4^n + 6n - 1$.

When $n = 0$, $4^n + 6n - 1 = 0$ and, as this is a multiple of 9, $P(0)$ is true, and we have the basis for induction.

So suppose that $P(k)$ is true for some integer $k \geq 0$; that is $4^k + 6k - 1 = 9m$ for some integer m. Then,

$$\begin{aligned} 4^{k+1} + 6(k+1) - 1 &= 4 \times 4^k + 6(k+1) - 1 \\ &= 4(9m - 6k + 1) + 6(k+1) - 1 \\ &= 36m - 18k + 9 \\ &= 9(4m - 2k + 1), \end{aligned}$$

showing that 9 divides $4^{k+1} + 6(k+1) - 1$, which verifies $P(k+1)$, and completes the induction step.

The result follows by Mathematical Induction.

4 The general result illustrated is that 36 divides $10^n + 8$ for all $n \geq 2$.

The result is true for $n = 2$ since $108 = 3 \times 36$, which is the basis for the induction.

For the induction step suppose that 36 divides $10^k + 8$, for some integer $k \geq 2$. To be precise, suppose that $10^k + 8 = 36r$ for some integer r. Then,

$$\begin{aligned} 10^{k+1} + 8 &= 10 \times 10^k + 8 \\ &= 10(36r - 8) + 8 \\ &= 36(10r) - 72 = 36(10r - 2), \end{aligned}$$

showing that $10^{k+1} + 8$ is divisible by 36, as required.

Hence by Mathematical Induction, 36 divides $10^n + 8$ for all integers $n \geq 2$.

5 There exist integers m and n such that $am + bn = 1$. Multiplying by c gives

$$m(ac) + (cn)b = c.$$

This expresses c as an integer combination of ac and b and so, by Problem 4.4 part (a), c is a multiple of $\gcd(ac, b)$. But, by definition, b is a multiple of $\gcd(ac, b)$ and so $\gcd(ac, b)$ is a common divisor of b and c and, in particular,

$$\gcd(ac, b) \leq \gcd(b, c).$$

On the other hand $\gcd(b, c)$ divides b and $\gcd(b, c)$ divides ac, (since it divides c), and so

$$\gcd(b, c) \leq \gcd(ac, b).$$

The equality follows.

6 (a) Let $\gcd(2a+b, a+b) = d$. Then, using property (e) of Theorem 4.2,

$$d \mid ((2a+b)-(a+b)), \text{ that is, } d \mid a$$

and

$$d \mid ((-1)(2a+b)+2(a+b)), \text{ that is, } d \mid b.$$

It follows that d divides $\gcd(a, b)$, and so $d = 1$.

(b) Let $\gcd(a+b, a-b) = d$. Then, using property (e) of Theorem 4.2,

$$d \mid ((a+b)+(a-b)), \text{ that is, } d \mid 2a$$

and

$$d \mid ((a+b)-(a-b)), \text{ that is, } d \mid 2b.$$

Hence

$$d \leq \gcd(2a, 2b) = 2\gcd(a, b) = 2,$$

as required.

$\gcd(2a, 2b) = 2\gcd(a, b)$ by Problem 4.4 part (d).

Section 5

1 Since we can write

$$\begin{aligned} 143 &= 8 \times 17 + 7 \\ 17 &= 2 \times 7 + 3 \\ 7 &= 2 \times 3 + 1 \\ 3 &= 3 \times 1 + 0, \end{aligned}$$

it follows that

$$\begin{aligned} \gcd(143, 17) &= 1 \\ &= 7 - 2 \times 3 \\ &= 7 - 2(17 - 2 \times 7) \\ &= -2 \times 17 + 5 \times 7 \\ &= -2 \times 17 + 5(143 - 8 \times 17) \\ &= 5 \times 143 - 42 \times 17. \end{aligned}$$

Therefore $x = -42$, $y = 5$ is one solution. The general solution is therefore

$$x = -42 + 143k, \quad y = 5 - 17k, \quad k \text{ any integer.}$$

The particular solution in which x takes its least positive value occurs when $k = 1$, and is $x = 101$, $y = -12$.

2 Since we can write

$$\begin{aligned} 158 &= 2 \times 57 + 44 \\ 57 &= 1 \times 44 + 13 \\ 44 &= 3 \times 13 + 5 \\ 13 &= 2 \times 5 + 3 \\ 5 &= 1 \times 3 + 2 \\ 3 &= 1 \times 2 + 1 \\ 2 &= 2 \times 1 + 0, \end{aligned}$$

it follows that

$$\begin{aligned}\gcd(158, 57) &= 1 \\ &= 3 - 1 \times 2 \\ &= 3 - (5 - 1 \times 3) = -1 \times 5 + 2 \times 3 \\ &= -1 \times 5 + 2(13 - 2 \times 5) = 2 \times 13 - 5 \times 5 \\ &= 2 \times 13 - 5(44 - 3 \times 13) = -5 \times 44 + 17 \times 13 \\ &= -5 \times 44 + 17(57 - 44) = 17 \times 57 - 22 \times 44 \\ &= 17 \times 57 - 22(158 - 2 \times 57) = -22 \times 158 + 61 \times 57.\end{aligned}$$

Therefore $x = -22$, $y = -61$ is one solution of the Diophantine equation $158x - 57y = 1$ and, multiplying by 7, $x = -154$, $y = -427$ is one solution of the equation $158x - 57y = 7$.

The general solution of the latter equation is therefore given by

$$x = -154 + 57k, \quad y = -427 + 158k, \quad k \text{ any integer.}$$

When $k \geq 3$, x and y are both positive, the least positive solution being $x = 17$, $y = 47$. The set of all positive solutions is

$$x = 17 + 57s, \quad y = 47 + 158s, \quad s \geq 0.$$

3 Suppose that x 2p coins, y 10p coins and z 50p coins are used. Then, since the total value is £6,

$$2x + 10y + 50z = 600.$$

Furthermore, as the total number of coins used is 100,

$$x + y + z = 100.$$

Substituting $2x = 200 - 2y - 2z$ into the former equation gives $200 + 8y + 48z = 600$, which simplifies to

$$y + 6z = 50.$$

One solution of this Diophantine equation may be spotted immediately, namely $y = 50$, $z = 0$. Hence the general solution is

$$y = 50 - 6k, \quad z = k, \quad k \in \mathbb{Z}.$$

Now as $x = 100 - y - z$, we seek non-negative solutions from

$$x = 50 + 5k, \quad y = 50 - 6k, \quad z = k, \quad k \in \mathbb{Z}.$$

Choosing $k \geq 0$ guarantees that x and z are non-negative and $k \leq 8$ keeps y non-negative. Hence there are nine solutions corresponding to $k = 0$ to 8 inclusive. These are given in the following table.

k	x	y	z
0	50	50	0
1	55	44	1
2	60	38	2
3	65	32	3
4	70	26	4
5	75	20	5
6	80	14	6
7	85	8	7
8	90	2	8

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